Regularizations of first order partial differential equations by generators of semigroups.

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Regularizations of first order partial differential equations by generators of semigroups

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Contents

Abstract

This thesis investigates the limiting behaviour of solutions to certain partial and pseudo differential equations. Included is a study of the notion of generalised solutions and particular examples, with emphasis on hyperbolic conservation laws. A probabilistic interpretation of some results is also presented.

It is known that the solution of initial-value problems for Burgers' equation

$$u_t - \epsilon u_{xx} + u u_x = 0,$$

($\epsilon > 0$) tends in the limit as $\epsilon \to 0$ to the solution of the hyperbolic conservation law

$$u_t + u(u)_x = 0,$$

subject to the same initial conditions. We find similar results for

$$u_t + \epsilon q(D)u + b \cdot \nabla u = 0,$$

and

$$u_t + b \cdot \nabla u = 0,$$

where $b \in \mathbb{R}$ and $q(D)$ is a pseudo-differential operator that generates a Feller semigroup.

We also study the equation

$$u_t + q(x, D)u + [q(x, D)]^{\frac{1}{2}} f(u) = 0,$$

where $q(x, D)$ generates a sub-Markovian semigroup.

Applications can be found in fluid dynamics.

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I would also like to thank my family, especially my mother for her encouragement and belief. This thesis is dedicated to the memory of my grandmother, "Mam".

Introduction

Hyperbolic conservation laws are central to the study of fluid dynamics, especially gas dynamics. Many present day classical results were obtained in the period between 1950 and 1970. These include results due to O. A. Oleinik [21] on viscosity solutions, i.e. approximations of solutions of hyperbolic conservation laws found by studying solutions of a parabolic regularization. A comprehensive presentation of the theory of hyperbolic conservation laws, due to C. Dafermos, is found in [5].

The simplest non-trivial problem is the initial-value problem

$$\text{(I.1)} \quad \begin{cases} u_t + \frac{1}{2}(u^2)_x = u_t + uu_x &= 0 \\ u(x,0) &= g(x), \end{cases}$$

where $x \in \mathbb{R}$, $t \geq 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a suitable function. One may also consider the more general problems

$$\text{(I.2)} \quad \begin{cases} u_t + \nabla \cdot f(u) &= 0 \\ u(x,0) &= g(x), \end{cases}$$

for $x \in \mathbb{R}^n$, $t \geq 0$ and given functions $f : \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$, or

$$\text{(I.3)} \quad \begin{cases} u_t + \nabla \tilde{f}(u) &= 0 \\ u(x,0) &= g(x), \end{cases}$$

where $x \in \mathbb{R}^n$, $t \geq 0$ and $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are given. In this case $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ must be vector-valued.

A viscosity solution to (I.1) is obtained by solving, for $\epsilon > 0$, the problem

$$\text{(I.4)} \quad \begin{cases} u_t - \epsilon \dfrac{\partial^2 u}{\partial x^2} + uu_x &= 0 \\ u(x,0) &= g(x), \end{cases}$$

and then proving that as $\epsilon \to 0$ the solutions u^ϵ to (I.4) converge to a solution of (I.1). Equation (I.4) and its higher dimensional analogue

$$\text{(I.5)} \quad u_t - \epsilon \triangle_n u + \sum_{j=1}^{n} uu_{x_j} = 0$$

are known as Burgers' equation. It is, however, well known that (I.1) does not always have a classical solution. The above procedure may therefore be a possible way of obtaining certain generalised solutions.

To a large extent the first chapter is devoted to the discussion of properties of problem (I.1) and various notions of generalised solutions are introduced along with the development of methods of obtaining such solutions. Most of this is well known, almost classical, material taken from L. C. Evans [7] and was included in the hope that an understanding of these results would give deeper insights into the more complicated situations.

In recent years an equation of the form

$$\text{(I.6)} \quad u_t + q(D)u + \nabla \cdot f(u) = 0$$

has been studied in detail in place of Burgers' equation. Here $q(D)$ is a certain pseudo-differential operator, e.g. $q(D) = -(-\triangle)^\alpha$, $0 < \alpha < 1$. Examples can be found in the work of P. Biler, T. Funaki and W. A. Woyczyński in [1] as well as P. Biler, G. Karch and W. A. Woyczyński [2], [3] and the references therein. In [3] it was suggested that one could take for $q(D)$ any generator of a Lévy process (subjected to some technical conditions). For example, let $q(D)$ be given on $C_0^\infty(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$ by

$$\text{(I.7)} \quad q(D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(\xi)\hat{u}(\xi) d\xi,$$

where the symbol $q : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function with Lévy-Khinchin representation

$$\text{(I.8)} \quad q(\xi) = c + ib\cdot\xi + \mathcal{Q}(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix\cdot\xi}{1+|x|^2}\right) \nu(dx).$$

Here $c \geq 0$, $b \in \mathbb{R}^n$, $\mathcal{Q}(\xi) = \sum_{j,l=1}^{n} q_{jl}\xi_j\xi_l \geq 0$ with $q_{jl} = q_{lj} \in \mathbb{R}$ and ν is a Borel measure on $\mathbb{R}^n \setminus \{0\}$ such that $\int_{\mathbb{R}^n\setminus\{0\}} \left(1 \wedge |x|^2\right) \nu(dx) < \infty$.

The original aim of this thesis was to answer the question of whether solutions to

$$\text{(I.9)} \quad \left\{ \begin{array}{rcl} u_t + \epsilon q(D)u + \nabla \cdot f(u) &=& 0 \\ u(x,0) &=& g(x) \end{array} \right.$$

will converge as $\epsilon \to 0$ to generalised solutions of (I.2). This, however, turned out to be too difficult to solve but partial answers were obtained as well as generalisations of results discussed in [3]. If one considers, instead of (I.9), a linear problem, i.e.

$$\text{(I.10)} \quad \begin{cases} u_t + \epsilon q(x,D)u + b \cdot \nabla u &= 0 \\ u(x,0) &= g(x) \end{cases}$$

where $b \in \mathbb{R}^n$, then one may use the fact that $-q(x, D)$ generates a Feller semigroup to prove that the solutions u^ϵ of (I.10) do indeed converge as $\epsilon \to 0$ to the solutions of the (linear) problem

$$\text{(I.11)} \quad \begin{cases} u_t + b \cdot \nabla u &= 0 \\ u(x,0) &= g(x). \end{cases}$$

Moreover, in this case the special structure of $-q(D)$ is not needed and in fact any generator, A, of a Feller semigroup will do. These results are discussed in detail in Chapter 2. Although they are straightforward to obtain, we know of no literary reference containing them. Chapter 2 ends with a discussion of how the limiting procedure works in the case of the classical nonlinear equation, i.e. Burgers' equation. This is largely from L. C. Evans [7]. A result due to P. Lax [20] is that the solution, u^ϵ, of Burgers equation can be represented as

$$\text{(I.12)} \quad u^\epsilon(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \left(\frac{x-y}{t} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{1}{2\epsilon}\int_{-\infty}^{y} g(z)dz} \right) dy}{\displaystyle\int_{-\infty}^{\infty} \left(e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{1}{2\epsilon}\int_{-\infty}^{y} g(z)dz} \right) dy}.$$

Denoting Brownian Motion by $(Y_t)_{t\geq 0}$ we may rewrite (I.12) as

$$\text{(I.13)} \quad u^\epsilon(x,t) = \frac{E^x \left(\frac{x-Y_t}{t} e^{-\frac{1}{2\epsilon}\int_{-\infty}^{Y_t} g(z)dz} \right)}{E^x \left(e^{-\frac{1}{2\epsilon}\int_{-\infty}^{Y_t} g(z)dz} \right)}.$$

It would be interesting to know whether (I.13) would make it possible to pass to the limit as $\epsilon \to 0$ when $(Y_t)_{t\geq 0}$ is substituted by the Feller process $(X_t)_{t\geq 0}$ generated by A. (This observation is due to N. Jacob). The structure of a generator, A, of a Feller semigroup (on $\mathbb{R}^n$) is well known provided some minimal assumptions are made on its domain $D(A)$. If, for example, $C_0^\infty(\mathbb{R}^n) \subset D(A)$ then A is, on $C_0^\infty(\mathbb{R}^n)$, a pseudo-differential operator

$$\text{(I.14)} \quad A = -q(x,D),$$

where

(I.15) $q(x,D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)d\xi$

and the symbol $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ is a measurable function such that $q(x,\cdot) : \mathbb{R}^n \longrightarrow \mathbb{C}$ is for all $x \in \mathbb{R}^n$ a continuous negative definite function. This result is due to Ph. Courrège [4]. W. Hoh and N. Jacob have, in many papers, given sufficient conditions that ensure $-q(x,D)$, given by (I.15) on $\mathcal{S}(\mathbb{R}^n)$, indeed extends to a generator of a Feller semigroup or an L^2-sub-Markovian semigroup. Particular references are W. Hoh [9] - [12] and N. Jacob [13] - [15] as well as the summaries [16] and [17].

In Chapter 3 the nonlinear term in (I.9) is modified and the problem

(I.16) $$\left\{ \begin{array}{rcl} u_t + q(x,D)u + [q(x,D)]^{\frac{1}{2}} f(u) & = & 0 \\ u(x,0) & = & g(x) \end{array} \right.$$

is studied, where in this case $-q(x,D)$ is the generator of a symmetric L^2-sub-Markovian semigroup. A fixed point argument yields the existence, locally in time, of a unique mild solution, i.e. for some $T > 0$ there exists

$$u \in L^\infty((0,T); L^2(\mathbb{R}^n))$$

such that

(I.17) $u(x,t) = T_t g(x) - \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(u(x,s))ds$

holds. This is a completely new result which extends to the case

(I.18) $u_t + q(x,D)u + [q(x,D)]^\alpha f(u) = 0$

for $0 < \alpha < 1$. Of special interest is the case when $||[q(x,D)]^\alpha u||_{L^2} \sim ||\nabla u||_{L^2}$ because this might lead to a way of studying

(I.19) $u_t + q(x,D)u + b \cdot \nabla f(u) = 0.$

However, for this one would need a control on the commutator $[[q(x,D)]^\alpha, \frac{\partial}{\partial x_j}]$ such that it would still be possible to estimate the integrals in (I.17).

Chapter 1

Preliminary Results

This chapter presents a collection of the preliminary results needed in further chapters as well as a discussion of the main concepts used such as weak solutions and their formulation and pseudo-differential operators as generators of semigroups.

1.1 The Method of Characteristics

Suppose we are given a first order partial differential equation defined on some subset of $\mathbb{R}^n$ involving a function with known values on the boundary. Below is an outline of the method of characteristics as described in [7] (§ 3.2.1). The aim, briefly, is to find certain curves passing through the boundary along which we are able to calculate the solution by referring back to the value at the intersection of the curve and the boundary.

Let Du represent the gradient vector of u, i.e. $Du = \nabla u = (u_{x_1}, \ldots, u_{x_n})$ where $u = u(x)$ and $x = (x_1, \ldots x_n)$ is a point in $\mathbb{R}^n$. We study, in $\mathbb{R}^n$, the first order partial differential equation

(1.1) $F(Du, u, x) = 0$ in U

with boundary condition

(1.2) $u = g$ on Γ

where U is an open subset of $\mathbb{R}^n$ with boundary ∂U and $\Gamma \subset \partial U$ and $g : \Gamma \to \mathbb{R}$ are given. Assume F and g are smooth functions (i.e. functions in C^k for $k \in \mathbb{N}$ large enough to ensure sufficient differentiability properties of the functions for the problem under consideration). We will use the notation $z := u(x)$; $p := Du(x)$. We now proceed to find the afore mentioned curves and begin by supposing such a curve to be defined parametrically by $\mathbf{x}(s) = (x^1(s), \ldots, x^n(s))$, for s in some subinterval of $\mathbb{R}$. Let $u = u(x_1, \ldots, x_n)$ be a solution of (1.1). Define

$$z(s) := u(\mathbf{x}(s))$$

$$\mathbf{p}(s) := Du(\mathbf{x}(s))$$

where $\mathbf{p}(s) = (p^1(s), \ldots, p^n(s))$, $p^i(s) = u_{x_i}(\mathbf{x}(s))$, $(i = 1, \ldots, n)$. The following ordinary differential equations are satisfied:

$$(1.3)\quad \begin{cases} \dot{\mathbf{p}}(s) &= -D_xF(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_zF(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s), \\ \dot{z}(s) &= D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s), \\ \dot{\mathbf{x}}(s) &= D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)). \end{cases}$$

The above is a system of $2n+1$ first order ordinary differential equations which are known as the *characteristic equations* of the partial differential equation (1.1). We will call the projection of the solutions of the characteristic equations onto the region $U \subset \mathbb{R}^n$ the *characteristic curves.*

Example 1.1. ([7], § 3.2.2 b) *Let $n = 2$ and consider the partial differential equation*

$$(1.4)\quad F(Du, u, x) = \mathbf{b}(x, u(x)) \cdot Du(x) + c(x, u) = 0, \quad x \in U,$$

or

$$(1.4')\quad F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(x, z),$$

whence $D_pF = \mathbf{b}(x, z)$. Then the third and second equations from (1.3) *become*

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s))$$

and

$$\begin{aligned} \dot{z}(s) &= \mathbf{b}(\mathbf{x}(s), z(s)) \cdot \mathbf{p}(s) \\ &= -c(\mathbf{x}(s), z(s)) \end{aligned}$$

respectively, the last equality following from (1.4′). *(The first equation from* (1.3) *is not needed in this case since all the information required to construct a solution can be obtained from the second and third equations).*

When $\mathbf{b} = (a, b)$ for $a, b \in \mathbb{R}$ and $c(x, u) = 0$ equation (1.4) *becomes*

$$au_{x_1} + bu_{x_2} = 0$$

and the characteristic equations are

$$\frac{dx_1}{ds} = a; \ \frac{dx_2}{ds} = b; \ \frac{dz}{ds} = 0$$

which give

$$x_1(s) = as + a_0;\; x_2(s) = bs + b_0;\; z = c_0$$

for constants $a_0, b_0, c_0 \in \mathbb{R}$. *The characteristic curves are then given by*

$$s \mapsto \begin{pmatrix} a \\ b \end{pmatrix} s + \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

along which z *and hence the solution* u *is constant. The value of this constant is found by tracing the curve back to the boundary where* u *is known.*

Example 1.2. *If we take* $\mathbf{b}(x,u) = (1,u)$ *and* $c(x,u) = 0$ *in* (1.4) *the equation becomes*

$$u_{x_1} + uu_{x_2} = 0$$

and the corresponding characteristic equations are

$$\frac{dx_1}{ds} = 1;\; \frac{dx_2}{ds} = z;\; \frac{dz}{ds} = 0.$$

Thus

$$x_1 = s + A_0,\; x_2 = z_0 s + B_0 \text{ and } z = z_0.$$

Relabel x_1, x_2 *as* t, x *respectively. To see how the solution propagates rewrite the characteristic equations in the form*

$$t = \frac{x}{z_0} - \frac{B_0}{z_0} + A_0$$

and note that the solution is constant along this curve. If an initial condition $u(x,0) = g(x)$ *is given then the solution on the curve passing through* x_0 *has the value* $z_0 = g(x_0)$. *Thus the solution along*

$$t = \frac{x}{g(x_0)} - \frac{x_0}{g(x_0)}$$

has the value $g(x_0)$. *The entire solution* u *is found as* x_0 *varies along the initial line* $t = 0$.

Below are the characteristic curves in the region $t \geq 0$ *for some particular choices of* $g(x)$.

(a) For the initial condition

$$(g.1) \qquad g_1(x) = \begin{cases} 1 & , \quad x \leq 0 \\ x+1 & , \quad 0 \leq x \leq 1 \\ 2 & , \quad x \geq 1 \end{cases}$$

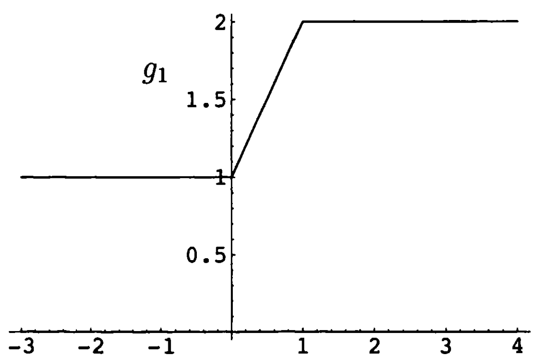

Fig.1.1: g_1

the characteristic curves are shown in the following diagram:

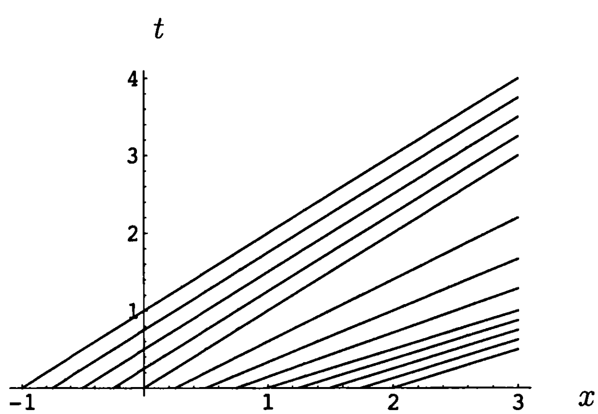

Fig.1.2: Characteristic curves when $g = g_1$

The solution is given by

$$u(x,t) = \begin{cases} 1 & , \quad x \leq t \\ \frac{1+x}{1+t} & , \quad t \leq x \leq 2t+1 \\ 2 & , \quad x \geq 2t+1 \end{cases}$$

which is shown below:

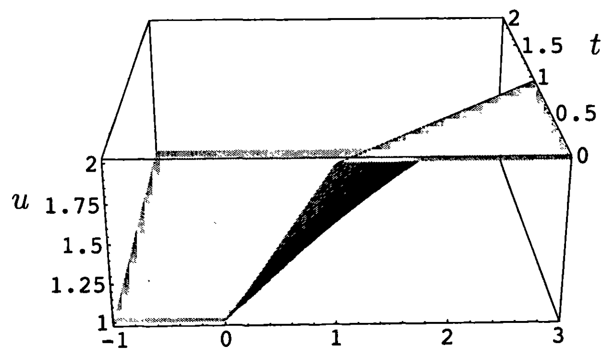

Fig.1.3: Solution when $g = g_1$

(b) For the initial condition

$$(g.2) \qquad g_2(x) = \begin{cases} 1 & , \quad x \leq 0 \\ 1 - x & , \quad 0 \leq x \leq 1 \\ 0 & , \quad x \geq 1 \end{cases}$$

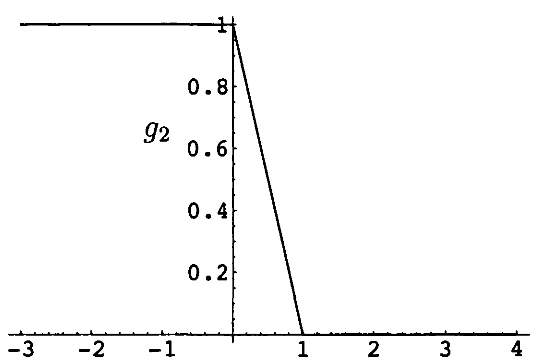

Fig.1.4: g_2

the characteristic curves are as follows:

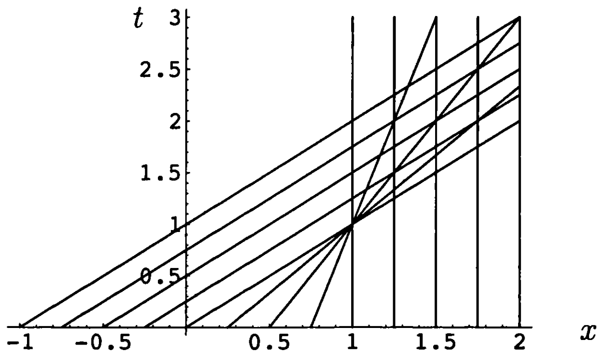

Fig.1.5: Characteristic curves when $g = g_2$

In the region $0 \leq t \leq 1$ the solution at a point is found by tracing the curve passing through the point back to the boundary. In the region $t > 1$, $1 < x < t$, a solution cannot be defined by our method since each point is the intersection of three characteristic curves. In fact, infinitely many curves pass through the point $(1, 1)$.

(c) For the initial condition

$$(g.3) \qquad g_3(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x \geq 0 \end{cases}$$

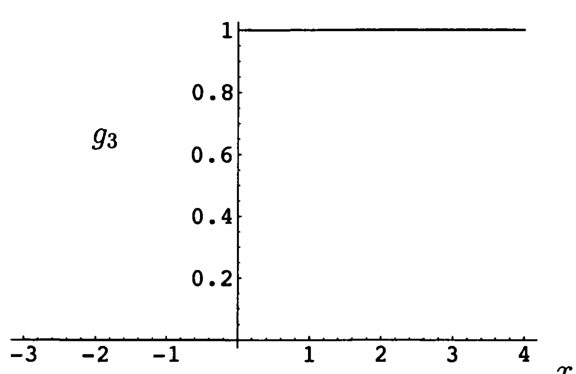

Fig.1.6: g_3

the characteristic curves are shown in the following diagram:

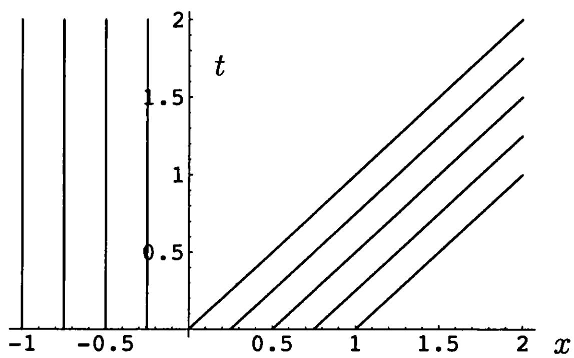

Fig.1.7: Characteristic curves when $g = g_3$

In the region $0 < x < t$ *there are no characteristic curves and we cannot define a solution by our method.*

(d) *Finally, for the initial condition*

$$(g.4) \qquad g_4(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad 0 \le x \le 1 \\ 0 & , \quad x \ge 1 \end{cases}$$

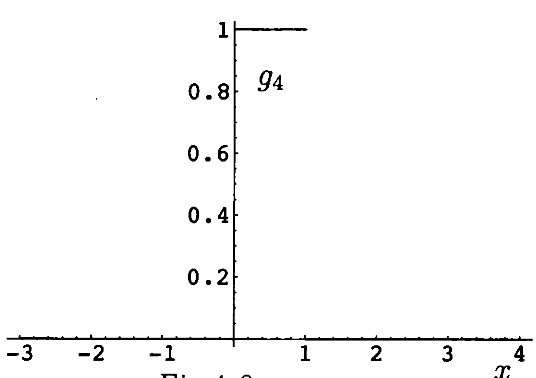

Fig.1.8: g_4

the corresponding characteristic curves are:

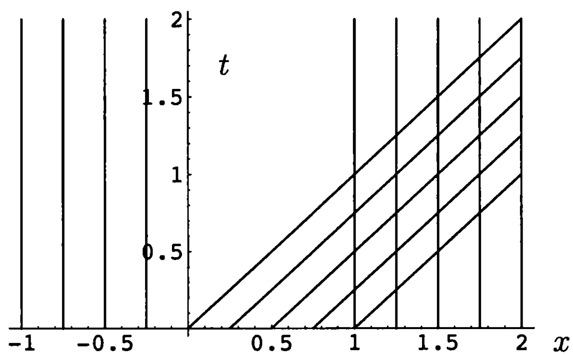

Fig.1.9: Characteristic curves when $g = g_4$

There are regions in which characteristic curves cross and also where there are no characteristic curves at all. Thus it is again not possible to define a solution for all $t > 0$ by our method.

Remark 1.3. *Note that although it is assumed in section* 1.1 *that the initial condition g is a smooth function, Example* 1.2 *uses non-smooth functions to more easily illustrate the crossing of characteristic curves and this does not affect the theory.*

1.2 Weak Solutions

A partial differential equation of order k is an equation involving an unknown function and its partial derivatives of order up to and including k. We say that a given problem for such an equation is **well-posed** if it has a unique solution that depends continuously on the given data. (Note that the notion of well-posed problems depends on the particular function or distribution space in which the equation is being considered). If the equation is of order k then we call a solution that is at least k times differentiable a **classical** solution. Solving a partial differential equation in the classical sense means finding, or showing the existence of, a classical solution of a well-posed problem (but there do exist problems that are not well-posed yet have several solutions and some that have no solution at all). This, however, is often not achievable and we must therefore introduce the notion of "weak solutions". This increases the class of solutions to include less regular functions while still keeping the problem well-posed. The precise definition of weak solution depends on the particular partial differential equation under consideration. ([7], §1.3.1).

Recall that in Example 1.2 it was found that for certain choices of the initial condition $g(x)$ the method of characteristics breaks down. In particular, for $(g.2)$ there were regions in which the characteristic curves crossed and there was no possibilty of singling out a unique value in order to define a solution u. For $(g.3)$ there were regions containing no characteristic curves and hence no possibility of defining a solution. We must therefore look for some sort of weak solution if we are to solve

$$(1.5) \quad u_t + uu_x = 0.$$

In fact, for the more general problem

$$(1.6) \quad \begin{cases} u_t + F(u)_x &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u &= g \text{ on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a given function, we have:

Definition 1.4. *We say that $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an* integral solution *of* (1.6) *if*

$$(1.7) \quad \int_0^\infty \int_{-\infty}^\infty (uv_t + F(u)v_x)dxdt + \int_{-\infty}^\infty gvdx|_{t=0} = 0$$

for each smooth function $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with compact support.

Remark 1.5. *We call such functions v test functions.*

Remark 1.6. *Here we follow* [7](§3.4.1*a*) *by calling this particular type of weak solution an "integral solution" but it is not the standard terminology.*

As noted we cannot in general find smooth solutions to (1.5) but the above definition does admit functions with curves of discontinuity called shock waves. Their construction is detailed below, following [7] (§3.4.1):

Let u be an integral solution of (1.5). Suppose that in some open region $V \subset \mathbb{R} \times (0, \infty)$, u is, together with its first derivatives, uniformly continuous except possibly along a smooth curve $\mathcal{C}$.

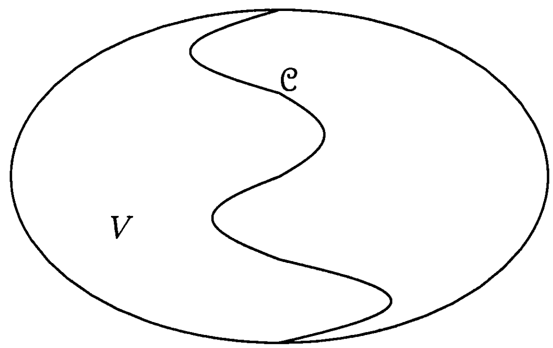

Fig.1.10: Region V and curve $\mathcal{C}$

Using integration by parts it can be shown that in V

$$u_t + F(u)_x$$

holds on each side of $\mathcal{C}$ and that

$$(F(u_l) - F(u_r))\nu^1 + (u_l - u_r)\nu^2 = 0 \tag{1.8}$$

along $\mathcal{C}$. Here $\nu = (\nu^1, \nu^2)$ is the unit normal of $\mathcal{C}$ in the direction from the region of V on the left of $\mathcal{C}$ to that on the right and u_l, u_r are the limits of u from the left and right respectively. If we represent $\mathcal{C}$ parametrically by $\{(x,t) : x = s(t)\}$ where $s(\cdot) : [0, \infty) \to \mathbb{R}$ is a smooth function we can take $\nu = (\nu^1, \nu^2) = (1 + \dot{s}^2)^{-\frac{1}{2}}(1, -\dot{s})$ and then (1.8) becomes

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r). \tag{1.9}$$

Denoting the jumps $u_l - u_r$ in u and $F(u_l) - F(u_r)$ in $F(u)$ across $\mathcal{C}$ by $[[u]]$ and $[[F(u)]]$ respectively and the speed $\dot{s}$ of $\mathcal{C}$ by σ we can rewrite (1.9) as

$$[[F(u)]] = \sigma[[u]] \qquad \text{(Rankine-Hugoniot condition).} \tag{1.10}$$

To ensure uniqueness we refine our class of solutions by considering only piecewise-smooth integral solutions u that have the following property: *If $(\tilde{x}, \tilde{t})$ is a point on a characteristic curve, $\tilde{l}$, corresponding to u that is not a point on any other characteristic curve then all points (x,t) on*

$\tilde{l}$ such that $t < \tilde{t}$ are also not on any other characteristic curve corresponding to u (i.e. the section of $\tilde{l}$ with values of t less than $\tilde{t}$ does not intersect any other characteristic curves).

For example, Fig.1.5 displays such a situation. Then suppose that on a curve $\mathcal{C}$ of discontinuities of u there is a point where two characteristic curves from either side of $\mathcal{C}$ meet and at which u_l and u_r are distinct left and right limits of u respectively. The equation can then be rewritten in the form

$$u_t + F'(u)u_x = 0$$

on either side of $\mathcal{C}$. The equations for the characteristic curves can then be found from

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s))$$

(recall the third equation of (1.3) in §1.1), where in this case $\mathbf{b}(\mathbf{x}(s), z(s)) = (1, F'(z))$. Therefore we have

$$\frac{dt}{ds} = 1; \ \frac{dx}{ds} = F'(z),$$

hence the characteristic curves are given by

$$s \mapsto (F'(z^0)s + x^0, s), \ \ s \geq 0$$

where x^0, z^0 are constants and $z^0 = g(x^0)$ for initial condition g. Thus the characteristic curves for (1.6) are

$$s \mapsto (F'(g(x^0))s + x^0, s), \ \ s \geq 0.$$

From this we deduce that along $\mathcal{C}$

$$F'(u_l) > \sigma > F'(u_r) \qquad \text{(entropy inequalities)}. \tag{1.11}$$

Remark 1.7. *If F is uniformly convex* (1.11) *reduces to*

$$u_l > u_r.$$

Definition 1.8. *A curve of discontinuity of a solution of* (1.6) *satisfying both the Rankine-Hugoniot condition and the entropy inequalities is called a* **shock wave**.

This can be used to construct integral solutions as follows: Consider the region $t \geq 0$ and recall Example 1.2 from §1.1.

When the initial condition is

$$g_2(x) = \begin{cases} 1 & , \quad x \leq 0 \\ 1 - x & , \quad 0 \leq x \leq 1 \\ 0 & , \quad x \geq 1 \end{cases}$$

we have a solution in the region $0 \leq t < 1$ given by

$$\tilde{u}(x,t) = \begin{cases} 1 & , \quad x \leq t \\ \frac{1-x}{1-t} & , \quad t \leq x \leq 1 \\ 0 & , \quad x \geq 1 \end{cases}$$

The curve $s(t) := \frac{1+t}{2}$ satisfies the Rankine-Hugoniot condition and so we can extend the above solution to the region $t \geq 1$ with s as a curve of discontinuity. The shock is shown in the following diagram:

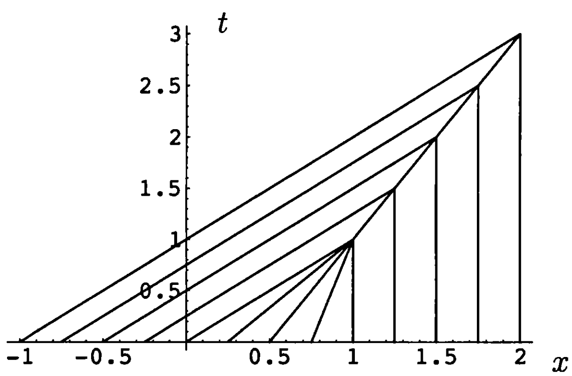

Fig.1.11: Shock wave when $g = g_2$

The complete solution is given by

$$u(x,t) = \begin{cases} 1 & , \quad x \leq t, \ t < 1 \\ \frac{1-x}{1-t} & , \quad t \leq x \leq 1, \ t < 1 \\ 0 & , \quad x \geq 1, \ t < 1 \\ 1 & , \quad x < s(t), \ t \geq 1 \\ 0 & , \quad x > s(t), \ t \geq 1 \end{cases}$$

which is shown in Fig 1.12:

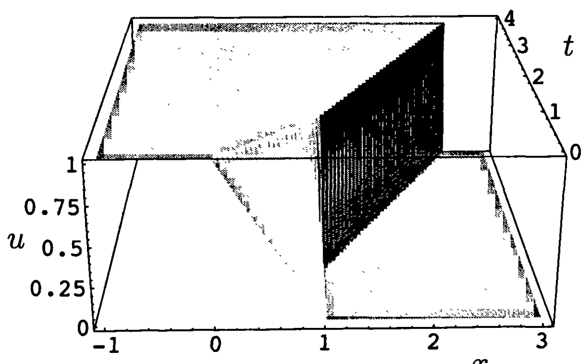

Fig.1.12: Integral solution when $g = g_2$

For the initial condition

$$g_3(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x \geq 0 \end{cases}$$

we have a solution given by

$$u(x,t) = \begin{cases} 0 & , \quad x < 0 \\ \frac{x}{t} & , \quad 0 < x < t \\ 1 & , \quad x > t \end{cases}$$

which is shown below:

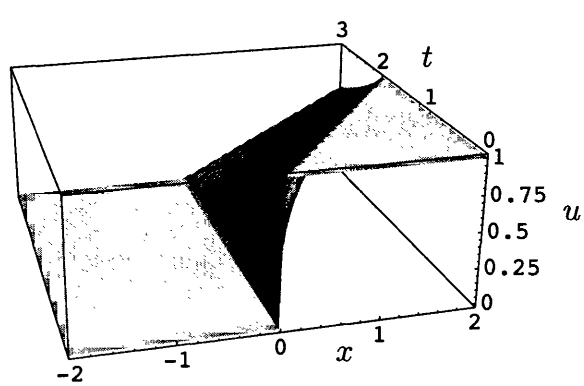

Fig.1.13: Integral solution when $g = g_3$

For the initial condition

$$g_4(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad 0 \leq x \leq 1 \\ 0 & , \quad x > 1 \end{cases}$$

we have a solution given by

$$u(x,t) = \begin{cases} 0 & , \quad x < 0,\ t \leq 2 \\ \frac{x}{t} & , \quad 0 < x < t,\ t \leq 2 \\ 1 & , \quad t < x < 1 + \frac{t}{2},\ t \leq 2 \\ 0 & , \quad x > \\ 0 & , \quad x > 1 + \frac{t}{2},\ t \leq 2 \\ 0 & , \quad x < 0,\ t \geq 2 \\ \frac{x}{t} & , \quad 0 < x < (2t)^{\frac{1}{2}},\ t \geq 2 \\ 0 & , \quad x > (2t)^{\frac{1}{2}},\ t \geq 2 \end{cases}$$

which is shown in the following diagram:

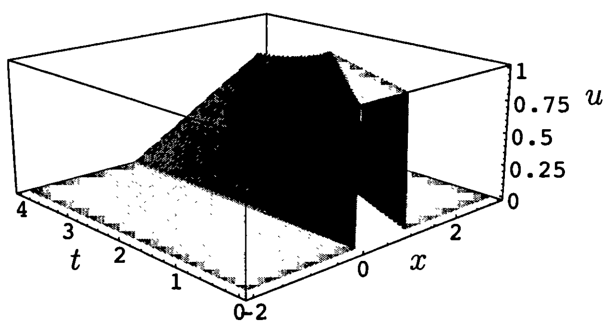

Fig.1.14: Integral solution when $g = g_4$

We now find a formula for integral solutions of (1.6): Following [7] (§3.3 and §3.4) we begin by first looking at the following initial-value problem for the Hamilton-Jacobi equation

$$(1.12) \qquad \begin{cases} u_t + H(\nabla u) &= 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

for $H : \mathbb{R}^n \to \mathbb{R}$ (see [7], §3.3). We make the following assumptions:

(A1) The mapping $p \mapsto H(p)$ is convex;

(A2) $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$;

(A3) $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous.

Also let L be the Legendre transform of H i.e.

$$L(p) = H^*(p) := \sup_{q\in\mathbb{R}^n} \{p \cdot q - H(q)\} \quad (p \in \mathbb{R}^n)$$

Definition 1.9. *We say that a Lipschitz continuous function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a* generalized solution *of the initial-value problem* (1.12) *if*

(*i*) $u(x,0) = g(x),\ x \in \mathbb{R}^n$;

(*ii*) $u_t(x,t) + H(\nabla u(x,t)) = 0$ *for a.e.* $(x,t) \in \mathbb{R}^n \times (0,\infty)$

and

(*iii*) $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C(1+\frac{1}{t})|z|^2$ *for some constant* $C \geq 0$ *and all* $x, z \in \mathbb{R}^n, t > 0$.

Definition 1.10. *A* C^2 *convex function is called* uniformly convex *(with constant* $\theta > 0$*) if*

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $p, \xi \in \mathbb{R}^n$.

Definition 1.11. *A function* $g : \mathbb{R}^n \to \mathbb{R}$ *is called* semiconcave *if there exists a constant* C *such that*

$$g(x+z) - 2g(x) + g(x-z) \leq C|z|^2$$

for all $x, z \in \mathbb{R}^n$.

Under assumptions $(A1) - (A3)$ and supposing that H is C^2 we have:

Theorem 1.12. ([7], §3.3.2, Thm 8, p.135)*If either g is semiconcave or H is uniformly convex, then*

$$u(x,t) = \min_{y\in\mathbb{R}^n} \{tL(\frac{x-y}{t}) + g(y)\} \tag{1.13}$$

is the unique generalized solution of the initial-value problem (1.12). □

Remark 1.13. *The expression on the r.h.s. of* (1.13) *is called the* Hopf-Lax formula.

Now we return to the initial-value problem (1.6). We make the following assumptions:

(A4) F is smooth and uniformly convex, $F(0) = 0$;

(A5) $g \in L^\infty(\mathbb{R})$.

Now let L be the Legendre transform of F and define

$$h(x) := \int_0^x g(y)dy.$$

From the above discussion of solutions of the Hamilton-Jacobi equation we know that

$$w(x,t) := \min_{y\in\mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\}$$

($x \in \mathbb{R}, t > 0$) is the unique generalized solution of

$$(1.14) \qquad \begin{cases} w_t + F(w_x) &= 0 \text{ in } \mathbb{R} \times (0,\infty) \\ w &= h \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

If w is smooth we can differentiate (1.12) w.r.t. x to get

$$\begin{cases} w_{xt} + F(w_x)_x &= 0 \text{ in } \mathbb{R} \times (0,\infty) \\ w_x &= g \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Hence w_x solves (1.6). However w is not, in general, smooth but is differentiable a.e. This suggests

$$(1.15) \quad u(x,t) := \tfrac{\partial}{\partial x}\left[\min_{y\in R}\left\{tL\left(\tfrac{x-y}{t}\right) + h(y)\right\}\right]$$

is a candidate for an integral solution of (1.6). Set $G := (F')^{-1}$. Then under assumptions $(A4)$ and $(A5)$ we have the following:

Theorem 1.14. ([7], §3.4.2, Thm 1)

(*i*) *For each time $t > 0$, there exists for all but at most countably many values of $x \in \mathbb{R}$ a unique point $\tilde{y}(x,t)$ such that*

$$\min_{y\in\mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\} = tL\left(\frac{x-\tilde{y}(x,t)}{t}\right) + h(\tilde{y}(x,t)).$$

(*ii*) *The mapping $x \mapsto \tilde{y}(x,t)$ is nondecreasing.*

(*iii*) *For each time $t > 0$, the function u defined by* (1.15) *is*

$$(1.16)\ u(x,t) = G\left(\frac{x-\tilde{y}(x,t)}{t}\right)$$

for a.e. $(x,t) \in \mathbb{R} \times (0,\infty)$.

□

Remark 1.15. *Equation* (1.16) *is called the* Lax-Oleinik formula *for the solution of* (1.6).

Finally we arrive at:

Theorem 1.16. ([7], §3.4.2 Thm 2) *Under assumptions* ($A4$) *and* ($A5$) *the function defined by* (1.16) *is an integral solution of* (1.6). □

Example 1.17. *In particular, if* $F(z) = \frac{z^2}{2}$ *we have that*

$$(1.17) \qquad \begin{cases} u_t + uu_x &= 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u &= g \text{ on } \mathbb{R} \times \{t=0\}. \end{cases}$$

has solution given by

$$u(x,t) = \frac{x-\tilde{y}(x,t)}{t}$$

where $\tilde{y}(x,t)$ *is the unique point at which for all* $t > 0$ *and for all but at most countably many* $x \in \mathbb{R}$

$$\min_{y\in\mathbb{R}}\left\{tL\left(\frac{x-y}{t}\right) + h(y)\right\} = tL\left(\frac{x-\tilde{y}(x,t)}{t}\right) + h(\tilde{y}(x,t))$$

(with $L = F^*$*).*

Uniqueness of the above solution is obtained via the following from [7] (§3.4.3):

Definition 1.18. *The inequality*

$$(1.18)\ u(x+z,t) - u(x,t) \leq \tfrac{C}{t}z,$$

where C *is a constant, is called the* entropy condition

Lemma 1.19. ([7], §3.4.3, Lemma, p.149). *Under assumptions* ($A4$) *and* ($A5$) *there exists a constant* C *such that the function* u *defined by the Lax-Oleinik formula satisfies the entropy condition* (1.18) *for all* $t > 0$ *and* $x, z \in \mathbb{R},\ z > 0$. □

Definition 1.20. *A function* $u \in L^\infty(\mathbb{R} \times (0,\infty))$ *is called an* entropy solution *of*

$$(1.6) \quad \begin{cases} u_t + F(u)_x &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u &= g \text{ on } \mathbb{R} \times t = 0 \end{cases}$$

if

(*i*) *u is an integral solution of* (1.6)

and

(*ii*) $u(x+z,t) - u(x,t) \leq C(1 + \frac{1}{t})z$

for some constant $C \geq 0$ and a.e. $x, z \in \mathbb{R}$, $t > 0$, with $z > 0$.

Theorem 1.21. ([7], §3.4.3 b, Thm 3). *If F is convex and smooth then there exists, up to a set of measure zero, at most one entropy solution of* (1.6). □

1.3 Fourier Analysis

The Fourier transform is a very useful tool within the theory of partial differential equations. Here are some elementary Fourier Analysis results from [16]:

Definition 1.22. *The* Schwartz space $\mathcal{S}(\mathbb{R}^n)$ *consists of all functions $u \in C^\infty(\mathbb{R}^n)$ such that for all $m_1, m_2 \in \mathbb{N}_0$*

$$p_{m_1,m_2}(u) := \sup_{x \in \mathbb{R}^n} ((1 + |x|^2)^{\frac{m_1}{2}} \sum_{|\alpha| \leq m_2} |\partial^\alpha u(x)|) < \infty.$$

$(p_{m_1,m_2})_{m_1,m_2 \in \mathbb{N}_0}$ is a family of separating seminorms. An equivalent family of seminorms is given by

$$p_{\alpha,\beta}(u) := \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha u(x)|, \alpha, \beta \in \mathbb{N}_0^n.$$

$\mathcal{S}(\mathbb{R}^n)$ equipped with the topology generated by one of these families is a Fréchet space (i.e. a locally convex metrizable vector space such that the metric is complete).

Definition 1.23. *The* convolution *of two functions $u, v \in \mathcal{S}(\mathbb{R}^n)$ is defined by*

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy.$$

For $u, v \in \mathcal{S}(\mathbb{R}^n)$ we have

(*i*) $u \in \bigcap_{p \geq 1} L^p(\mathbb{R}^n)$;

(*ii*) $u \cdot v \in \mathcal{S}(\mathbb{R}^n)$;

(*iii*) $u * v \in \mathcal{S}(\mathbb{R}^n)$;

(*iv*) $\phi \cdot u \in \mathcal{S}(\mathbb{R}^n)$ for any function $\phi \in C^\infty(\mathbb{R}^n)$ that, along with its partial derivatives, is polynomially bounded.

Theorem 1.24. ([16], Corollary 2.6.1). *The space $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$. The space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and in $C_\infty(\mathbb{R}^n)$.* □

Definition 1.25. *For $u \in \mathcal{S}(\mathbb{R}^n)$ we define the* Fourier transfrom F *of u by*

$$(1.19)\quad (Fu)(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx$$

and the inverse Fourier transform

$$(1.20)\quad (F^{-1}u)(\eta) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{iy\cdot\eta} u(y)dy.$$

Theorem 1.26. ([16], Thm 3.1.2, Thm 3.1.6). *Both F and F^{-1} are continuous linear operators from $\mathcal{S}(\mathbb{R}^n)$ into itself.* □

The name inverse Fourier transform is justified since on $\mathcal{S}(\mathbb{R}^n)$ it holds that $F \circ F^{-1} = F^{-1} \circ F = id$.

Remark 1.27. *The Fourier transform of u is also denoted by $\hat{u}$.*

Theorem 1.28. ([16], Lemma 3.1.9).

A. Let $T_a : \mathbb{R}^n \to \mathbb{R}^n$ be the translation operator $T_a x = a + x$, $a \in \mathbb{R}^n$. For $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(1.21)\quad (u \circ T_a)^\wedge(\xi) = e^{ia\cdot\xi}\hat{u}(\xi).$$

B. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear mapping. For $u \in \mathcal{S}(\mathbb{R}^n)$ it follows that $u \circ T$ is also an element of $\mathcal{S}(\mathbb{R}^n)$ and

$$(1.22)\quad (u \circ T)^\wedge(\xi) = \frac{1}{|detT|}\hat{u} \circ (T^{-1})^t(\xi).$$

In particular, for the reflection $Sx = -x$ we have

$$(1.23)\quad (u \circ S)^\wedge(\xi) = \hat{u}(-\xi)$$

and for the homothetic mapping $H_\lambda(x) = \lambda x$ we have

$$(1.24)\quad (u \circ H_\lambda)^\wedge(\xi) = \lambda^{-n}\hat{u}\left(\frac{\xi}{\lambda}\right),\ \lambda > 0.$$

C. For $u \in \mathcal{S}(\mathbb{R}^n)$ we have

(1.25) $\bar{\hat{u}}(\xi) = F^{-1}(\bar{u})(\xi)$. □

Theorem 1.29. ([16], Thm 3.1.10). *For all $u \in \mathcal{S}(\mathbb{R}^n)$ we have*

(1.26) $||\hat{u}||_\infty \leq (2\pi)^{-\frac{n}{2}}||u||_{L^1}$

and

(1.27) $||u||_0 = ||\hat{u}||_0$. □

Remark 1.30. *Equality* (1.27) *is a first version of Plancherel's theorem. For all $u, v \in \mathcal{S}(\mathbb{R}^n)$ we get, via polarisation,*

(1.28) $(u, v)_0 = (\hat{u}, \hat{v})_0$.

Theorem 1.31 (Convolution Theorem). ([16], Thm 3.1.12). *Let $u, v \in \mathcal{S}(\mathbb{R}^n)$. Then*

(i) $(u \cdot v)^\wedge(\xi) = (2\pi)^{\frac{-n}{2}}(\hat{u} * \hat{v})(\xi)$ *and*

(ii) $(u * v)^\wedge(\xi) = (2\pi)^{\frac{n}{2}}\hat{u}(\xi) \cdot \hat{v}(\xi)$. □

The Fourier transform can be extended from $\mathcal{S}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Let $u \in L^1(\mathbb{R}^n)$ and set

(1.29) $\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx.$

This integral is well defined since $x \mapsto e^{-ix\cdot\xi}u(x)$ is an element of $L^1(\mathbb{R}^n)$. We call (1.29) the Fourier transform of u in $L^1(\mathbb{R}^n)$ and note that since $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ it is an extension of the Fourier transform in $\mathcal{S}(\mathbb{R}^n)$.

Theorem 1.32. ([16], Thm 3.2.1). *The Fourier transform is a continuous linear operator from $L^1(\mathbb{R}^n)$ into $C_\infty(\mathbb{R}^n)$ and*

(1.30) $||\hat{u}||_\infty \leq (2\pi)^{-\frac{n}{2}}||u||_{L^1}$

holds for all $u \in L^1(\mathbb{R}^n)$. □

Theorem 1.28 also holds for $u \in L^1(\mathbb{R}^n)$ and we have the first part of the convolution theorem:

Theorem 1.33. ([16], Thm 3.2.4). *For $u, v \in L^1(\mathbb{R}^n)$ we have*

(1.31) $(u * v)^\wedge(\xi) = (2\pi)^{\frac{n}{2}}\hat{u}(\xi) \cdot \hat{v}(\xi)$. □

For $u \in L^1(\mathbb{R}^n)$ it is in general not possible to define an inverse Fourier transform for $\hat{u}$ by (1.20). For this reason we introduce the *Wiener algebra* on which it is possible to consider the inverse Fourier transform.

Definition 1.34. *The* Wiener algebra $A(\mathbb{R}^n)$ *is defined by*

$$A(\mathbb{R}^n) := \{u \in L^1(\mathbb{R}^n) : \hat{u} \in L^1(\mathbb{R}^n)\}.$$

Theorem 1.35. ([16], Corollary 3.2.12). *For $u \in A(\mathbb{R}^n)$ we have for almost all $x \in \mathbb{R}^n$*

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)d\xi.$$

□

Corollary 1.36. ([16], Corollary 3.2.13). *The Fourier transform is an injective mapping from $L^1(\mathbb{R}^n)$ into $C_\infty(\mathbb{R}^n)$.*

Now we consider the Fourier transform in $L^2(\mathbb{R}^n)$:
For $u \in L^2(\mathbb{R}^n)$ the integral

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx \tag{1.32}$$

does not in general converge and therefore cannot be used to define the Fourier transform of $u \in L^2(\mathbb{R}^n)$. However we have the following, via Plancherel's theorem:

Theorem 1.37. ([16], Thm 3.2.18). *The Fourier transform as it is defined on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ has an extension to $L^2(\mathbb{R}^n)$. This extension is an isometry on $L^2(\mathbb{R}^n)$ which is bijective and has a continuous inverse.*

□

Remark 1.38. *We denote the Fourier transform of $u \in L^2(\mathbb{R}^n)$ as it is obtained from Theorem 1.37 by Fu, or $\hat{u}$, as usual.*

For $\theta \in [0,1]$ set $p = \frac{2}{1+\theta}$ and $q = \frac{2}{1-\theta}$. The Fourier transform extends to a continuous linear mapping from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with norm less than or equal to $(2\pi)^{-n\frac{\theta}{2}}$.

Definition 1.39. (i) *Let $G \in \mathbb{R}^n$ be an open set. The topological dual space $\mathcal{D}'(G)$ of $C_0^\infty(G)$ is called the space of* distributions *on G.*

(ii) *The topological dual space $\mathcal{S}'(\mathbb{R}^n)$ of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is called the space of* tempered distributions. *It is the set of all elements of $\mathcal{D}'(\mathbb{R}^n)$ that have continuous extension to $\mathcal{S}(\mathbb{R}^n)$, hence $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.*

For $\mathcal{S}'(\mathbb{R}^n)$ we also have:

Theorem 1.40. ([16], p.46). *A. Functions u satisfying any of the following conditions are elements of $\mathcal{S}'(\mathbb{R}^n)$*

(i) $u \in L^p(\mathbb{R}^n),\ 1 \leq p \leq \infty$;

(ii) u is a polynomial;

(iii) u is a measurable, polynomially bounded function.

B. If $u \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and ϕ is a measurable, polynomially bounded function then $u\phi \in \mathcal{S}'(\mathbb{R}^n)$. □

Under the weak-$*$-topology $\mathcal{S}'(\mathbb{R}^n)$ is a topological vector space.

Theorem 1.41. ([16], Thm 2.6.9). *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. The convolution defined by*

$$(u * \phi)(x) := u(\phi(x - .)),$$

has at most polynomial growth. □

The Fourier transform can be extended from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ by duality.

Definition 1.42. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform $\hat{u}$ of u is defined by*

$$< \hat{u}, \phi > := < u, \hat{\phi} > \tag{1.33}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, where $<,>$ represents the duality pairing between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$.

Remark 1.43. *Note that $\hat{u} : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous (as a composition of continuous mappings). We again have Fu as an alternative notation to $\hat{u}$.*

Theorem 1.44. ([16], Thm 3.33). *The Fourier transform is a continuous linear operator from $\mathcal{S}'(\mathbb{R}^n)$ into itself which is bijective and has a continuous inverse F^{-1}.* □

1.4 Measure and Semigroup Theory

Now we collect, and state without proof, some relevant results from measure and semigroup theory. (See [16]).

Let G be a locally compact space and let $\mathcal{B}(G)$ denote its Borel σ-field. Measures on $\mathcal{B}(G)$ are called Borel measures.

Definition 1.45. *Let $(\Omega, \mathcal{A})$ be an arbitrary measure space and let μ be a measure on $(\Omega, \mathcal{A})$.*

(*i*) *The* total mass *of μ is defined by $\|\mu\| := \mu(\Omega)$.*

(*ii*) *The measure μ is called* bounded *if $\|\mu\| < \infty$. The set of all bounded measures on Ω is denoted by $\mathcal{M}_b^+(\Omega)$.*

(*iii*) *If $\|\mu\| = 1$ we call μ a probability measure. The set of all probability measure on Ω is denoted by $\mathcal{M}_b^1(\Omega)$.*

Definition 1.46. *Let G be a locally compact space. A* signed measure *on $(G, \mathcal{B}(G))$ is a mapping $\mu : \mathcal{B}(G) \mapsto \mathbb{R}$ such that we can write $\mu(A) = \mu_1(A) - \mu_2(A)$ for two measures $\mu_1, \mu_2 \in \mathcal{M}_b^+(G), A \in \mathcal{B}(G)$. The set of signed measures on G is denoted by $\mathcal{M}(G)$ and the set of bounded signed meaures on G is denoted by $\mathcal{M}_b(G)$.*

Definition 1.47. *Let $(\mu_\nu)_{\nu \in \mathbb{N}}$ be a sequence in $\mathcal{M}_b(G)$ and also let $\mu_0 \in \mathcal{M}_b(G)$.*

(*i*) *We say the sequence $(\mu_\nu)_{\nu \in \mathbb{N}}$* converges in norm *to μ_0 if, and only if,*

$$\lim_{\nu \to \infty} ||\mu_\nu - \mu_0|| = 0.$$

(*ii*) *The sequence $(\mu_\nu)_{\nu \in \mathbb{N}}$ is said to* converge weakly *to μ_0 if for all $u \in C_b(G)$ we have*

$$\lim_{\nu \to \infty} \int_G u(x) \mu_\nu(dx) = \int_G u(x) \mu_0(dx). \tag{1.34}$$

(*iii*) *We say the sequence $(\mu_\nu)_{\nu \in \mathbb{N}}$* converges with respect to C_∞ *to μ_0 if (1.34) holds only for all $u \in C_\infty(G)$.*

(*iv*) *We say that $(\mu_\nu)_{\nu \in \mathbb{N}}$* converges vaguely *to μ_0 if (1.34) holds only for all $u \in C_0(G)$.*

Remark 1.48. *The notion of vague convergence is also well-defined on $\mathcal{M}(G)$.*

The topology referring to the weak convergence of measures is known as the Bernoulli topology. Since $\mathcal{M}_b^+(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ the Fourier transform $\hat{\mu}$ of $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ is well-defined.

Theorem 1.49. ([16], Thm 3.5.1). *Let $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ be a bounded Borel measure. Its Fourier transform $\hat{\mu}$ is the uniformly continuous function on $\mathbb{R}^n$ given by*

$$\hat{\mu}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \mu(dx).$$

□

Theorem 1.50. ([16], Thm 3.5.2). *A. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. For $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ the image measure $T(\mu)$ satisfies*

(1.35) $[T(\mu)]^\wedge = \hat{\mu} \circ T^t.$

In particular, for the reflection $S : \mathbb{R}^n \to \mathbb{R}^n$, $Sx = -x$, we have

(1.36) $[S(\mu)]^\wedge = \bar{\hat{\mu}} = \hat{\mu} \circ S.$

B. For the translation $T_a : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + a$, we have

(1.37) $[T_a(\mu)]^\wedge = e^{-i\xi \cdot a}\hat{\mu}.$

C. For $\mu, \nu \in \mathcal{M}_b^+(\mathbb{R}^n)$ the convolution theorem holds, i.e.

(1.38) $(\mu * \nu)^\wedge = (2\pi)^{\frac{n}{2}}\hat{\mu} \cdot \hat{\nu}.$ □

Definition 1.51. *A family $(\mu_t)_{t\geq 0}$ of bounded Borel measures on $\mathbb{R}^n$ is called a* convolution semigroup *on $\mathbb{R}^n$ if*

(i) $\mu_t(\mathbb{R}^n) \leq 1$ *for all* $t \geq 0$;

(ii) $\mu_s * \mu_t = \mu_{t+s}$, $s, t \geq 0$ *and* $\mu_0 = \varepsilon_0$;

(iii) $\mu_t \to \varepsilon_0$ *vaguely as* $t \to 0$.

Here ε_0 is the Dirac measure.

Definition 1.52. (i) *A function $u : \mathbb{R}^n \mapsto \mathbb{C}$ is called* positive definite *if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ the matrix $(u(\xi^j - \xi^l))_{j,l=1,\ldots,k}$ is positive Hermitian, i.e. for all $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ we have*

$$\sum_{j,l=1}^{k} u(\xi^j - \xi^l)\lambda_j\bar{\lambda_l} \geq 0.$$

(ii) *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called* negative definite *if*

$$\psi(0) \geq 0$$

and

$$\xi \mapsto (2\pi)^{-\frac{n}{2}}e^{-t\psi(\xi)} \text{ is positive definite for all } t > 0.$$

Theorem 1.53. ([16], Lemma 3.5.4). *Let $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$. Then $\hat{\mu}$ is a positive definite function.* □

Theorem 1.54. ([16], Thm 3.6.16). *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. Then there exists a uniquely determined continuous negative definite funtion $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

(1.39) $\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$

for all $\xi \in \mathbb{R}^n$ *and* $t \geq 0$. *Conversely, given a continuous negative definite function* $\psi : \mathbb{R}^n \to \mathbb{C}$ *then there exists a unique convolution semigroup* $(\mu_t)_{t\geq 0}$ *on* $\mathbb{R}^n$ *such that* (1.39) *holds.* □

Example 1.55. ([16], Example 3.6.18). *Any non-negative symmetric quadratic form* $q : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ *is a continuous negative definite function. A convolution semigroup that has* q *as its corresponding negative definite function is called a* Gaussian semigroup.

Let $(X, \|\cdot\|_X)$ be a real or complex Banach space.

Definition 1.56. (*i*) *A one parameter family* $(T_t)_{t\geq 0}$ *of bounded linear operators* $T_t : X \to X$ *is called a* (one parameter) semigroup *of operators if* $T_0 = id$ *and* $T_{s+t} = T_s \circ T_t$ *for all* $s, t \geq 0$.

(*ii*) *We call* $(T_t)_{t\geq 0}$ strongly continuous *if*

$$\lim_{t\to 0} \|T_t u - u\|_X = 0$$

for all $u \in X$.

(*iii*) *The semigroup* $(T_t)_{t\geq 0}$ *is called a* contraction semigroup *if* $\|T_t\| \leq 1$ *for all* $t \geq 0$.

Definition 1.57. *Let* $(T_t)_{t\geq 0}$ *be a strongly continuous contraction semigroup on* $(C_\infty(\mathbb{R}^n), \|\cdot\|)$ *which is positivity preserving, i.e.* $u \geq 0$ *implies* $T_t u \geq 0$. *Then* $(T_t)_{t\geq 0}$ *is called a* Feller semigroup.

Definition 1.58. *Let* $(T_t)_{t\geq 0}$ *be a strongly continuous semigroup of operators on a Banach space* $(X, \|\cdot\|_X)$. *The* generator A *of* $(T_t)_{t\geq 0}$ *is defined by*

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t}$$

(strong limit) with domain

$$D(A) := \{u \in X : \lim_{t\to 0} \frac{T_t u - u}{t} \textit{ exists as a strong limit}\}.$$

Example 1.59. ([16], Example 4.1.12). *Let* $(\mu_t)_{t\geq 0}$ *be a convolution semigroup on* $\mathbb{R}^n$. *On the Banach space* $(C_\infty(\mathbb{R}^n), \|\cdot\|)$ *we define the operators*

$$T_t u(x) := \int_{\mathbb{R}^n} u(x-y)\mu_t(dy).$$

Then $(T_t)_{t\geq 0}$ *is a Feller semigroup. Also let* $\psi : \mathbb{R}^n \to \mathbb{C}$ *be the continuous negative definite function corresponding to the convolution semigroup*

$(\mu_t)_{t\geq 0}$. *Let* $u \in \mathcal{S}(\mathbb{R}^n)$ *and note that* $\mathcal{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ *is dense. It follows that*

$$\frac{T_t u - u}{t} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \frac{e^{-t\psi(\xi)} - 1}{t} \, \hat{u}(\xi) d\xi.$$

Then $\mathcal{S}(\mathbb{R}^n) \subset D(-\psi(D))$*, where* $-\psi(D)$ *is the generator of* $(T_t)_{t\geq 0}$ *and is given on* $\mathcal{S}(\mathbb{R}^n)$ *by*

$$\psi(D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi.$$

($D(-\psi(D))$ *denotes the domain of* $-\psi(D)$*). Note that* $T_t u(x)$ *can also be written in the form*

$$T_t u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\psi(\xi)t}\hat{u}(\xi)d\xi.$$

Definition 1.60. *A linear operator* $A : D(A) \to B_b(\mathbb{R}^n; \mathbb{R})$*,* $D(A) \subset B_b(\mathbb{R}^n; \mathbb{R})$*, satisfies the* positive maximum principle *if* $u \in D(A)$ *and* $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geq 0$ *implies* $Au(x_0) \leq 0$*. (*$B_b(\mathbb{R}^n; \mathbb{R})$ *is the set of bounded measurable mappings from* $\mathbb{R}^n$ *to* $\mathbb{R}$*).*

Theorem 1.61. ([16], Thm 4.5.3). *Let* A *be a linear operator on* $C_\infty(\mathbb{R}^n)$ *with domain* $D(A) \subset C_\infty(\mathbb{R}^n)$*.* A *is closable and its closure is the generator of a Feller semigroup if, and only if, the following hold:*

(i) $D(A) \subset C_\infty(\mathbb{R}^n)$ *is dense;*

(ii) $(A, D(A))$ *satisfies the positive maximum principle;*

(iii) $R(\lambda - A)$ *is dense in* $C_\infty(\mathbb{R}^n)$ *for some* $\lambda > 0$*.*

(Here $R(\lambda - A)$ *denotes the range of the operator* $\lambda - A$*).* □

Definition 1.62. *(i) Let* $(T_t)_{t\geq 0}$ *be a strongly continuous contraction semigroup on* $L^p(\mathbb{R}^n)$*,* $1 \leq p < \infty$*. If for* $u \in L^p(\mathbb{R}^n)$ *such that* $0 \leq u \leq 1$ *almost everywhere it follows that* $0 \leq T_t u \leq 1$ *almost everywhere then* $(T_t)_{t\geq 0}$ *is called a* sub-Markovian *semigroup on* $L^p(\mathbb{R}^n)$*,* $1 \leq p < \infty$*.*

(ii) Let $(T_t)_{t\geq 0}$ *be a strongly continuous contraction semigroup on* $L^p(\mathbb{R}^n)$*,* $1 \leq p < \infty$*, or on* $C_\infty(\mathbb{R}^n)$*. If for all* $u, v \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*, or* $u, v \in C_\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *respectively, we have*

$$(T_t u, v)_0 = (u, T_t v)_0$$

then $(T_t)_{t\geq 0}$ *is called a* symmetric *semigroup.*

Theorem 1.63. ([16], Lemma 4.1.14). *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on the Banach space $(X, ||\cdot||_X)$ with generator $(A, D(A))$, $D(A) \subset X$. A. For any $u \in X$ and $t \geq 0$ it follows that $\int_0^t T_s u ds \in D(A)$ and*

(1.40) $T_t u - u = A \int_0^t T_s u ds.$

B. For $u \in D(A)$ and $t \geq 0$ we have $T_t u \in D(A)$ and

(1.41) $\frac{d}{dt} T_t u = A T_t u = T_t A u.$

C. For $u \in D(A)$ and $t \geq 0$

(1.42) $T_t u - u = \int_0^t A T_s u ds = \int_0^t T_s A u ds$

holds. □

1.5 Solution Spaces

It is necessary to introduce certain function spaces in which solutions of the partial differential equations under consideration lie. Most noteworthy are Sobolev spaces and $H^{\psi,s}$-spaces.

Recall that F and $\wedge$ denote the Fourier transform.

1.5.1 Sobolev spaces

(See [7], §5.2). In the following let U be an open subset of $\mathbb{R}^n$and let α be a multiindex. For $u : U \to \mathbb{R}$, $x \in U$ define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(x).$$

Definition 1.64. *Suppose*

$$u, v \in L^p_{loc}(U) := \{u : U \to \mathbb{R} : v \in L^p(V) \text{ for each } u \subset\subset U\}$$

and α is a multiindex. We say that v is the α-th weak partial derivative of u, written

$$D^\alpha u = v$$

provided

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all functions $\phi \in C_0^\infty(U)$.

Definition 1.65. *Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The* Sobolev space

$$W^{k,p}(U)$$

consists of all locally summable functions $u : U \mapsto \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. If $u \in W^{k,p}(U)$ we define its norm to be

$$||u||_{W^{k,p}(U)} := \begin{cases} (\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \mathrm{ess\,sup_U}\, |\mathrm{D}^\alpha \mathrm{u}|, & p = \infty. \end{cases}$$

When $p = 2$ we write $W^{k,2}(U) = H^k(U)$. An alternative characterisation of the spaces $H^k(\mathbb{R}^n)$, using the Fourier transform, is as follows: Let k be a non-negative integer. We denote by $||u||_{H^k(\mathbb{R}^n)}$ the norm of $u \in W^{k,2}(\mathbb{R}^n)$.

(i) A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if, and only if,

$$(1 + |\cdot|^k)\hat{u} \in L^2(\mathbb{R}^n).$$

(ii) In addition, there exists a positive constant C such that

$$\frac{1}{C}||u||_{H^k(\mathbb{R}^n)} \leq ||(1 + |\cdot|^k)\hat{u}||_{L^2(\mathbb{R}^n)} \leq C||u||_{H^k(\mathbb{R}^n)}$$

for each $u \in H^k(\mathbb{R}^n)$.

Definition 1.66. *Assume $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$. Then $u \in H^s(\mathbb{R}^n)$ if $(1 + |\cdot|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For non-integer s we set*

$(H.1)$ $||u||_{H^s(\mathbb{R}^n)} := ||(1 + |\cdot|^s)\hat{u}||_{L^2(\mathbb{R}^n)}$.

Remark 1.67. *Since $(1 + |y|^s) \sim (1 + |y|^2)^{\frac{s}{2}}$ for $y \in \mathbb{R}$ (which follows from the inequalities $c_1(1+|y|^s) \leq (1+|y|^2)^{\frac{s}{2}} \leq c_2(1+|y|^s)$ for constants $c_1, c_2 \in \mathbb{R}$) an equivalent norm to $(H.1)$ is given by*

$(H.2)$ $||u||_{H^s(\mathbb{R}^n)} = ||(1 - \triangle)^{\frac{s}{2}}u||_{L^2(\mathbb{R}^n)} \sim ||(1 + |y|^s)\hat{u}||_{L^2(\mathbb{R}^n)}$.

It is often better to take this as the definition instead of $(H.1)$ because $(1 + |y|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}^n)$ is smooth if $\hat{u}$ is smooth.

The corresponding scalar product to the norm $(H.2)$ is given by

$$(u, v)_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |y|^2)^{\frac{s}{2}}\hat{u}(y)(1 + |y|^2)^{\frac{s}{2}}\bar{\hat{v}}(y)dy.$$

The notation $||\cdot||_s := ||\cdot||_{H^s}$ is often used. Note also that $||\cdot||_0 = ||\cdot||_{L^2}$.

1.5.2 $H^{\psi,s}$-Spaces

The class of spaces introduced above can be extended by replacing $\xi \to |\xi|^s$ in $(H.1)$ with a more general function. In the following, let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function and let $s \in \mathbb{R}$, $p \in [1, \infty]$.

Definition 1.68. *The space $B^s_{\psi,p}(\mathbb{R}^n)$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$(H.3)$ $||u||_{\psi,s,p} = ||(1 + |\psi(\cdot)|)^{\frac{s}{2}}\hat{u}(\cdot)||_{L^p(\mathbb{R}^n)} < \infty.$

For $p = 2$ we have the notation

$(H.4)$ $H^{\psi,s}(\mathbb{R}^n) := B^s_{\psi,2}(\mathbb{R}^n)$ and $||u||_{\psi,s} := ||u||_{\psi,s,2}.$

Theorem 1.69. ([16], Thm 3.10.3). *The space $B^s_{\psi,p}(\mathbb{R}^n)$ is a Banach space and in the sense of continuous embeddings we have*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{\psi,p}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Also, for $1 \le p < \infty$, $C_0^\infty(\mathbb{R}^n) \subset B^s_{\psi,p}(\mathbb{R}^n)$ is dense. □

Yet more spaces can be constructed from combinations of the above if we consider functions whose images are themselves functions.

Definition 1.70. *Let Z be a function space and let $1 \le p \le \infty$. The space*

$$L^p([0,T]; Z)$$

consists of functions $u : [0,T] \to Z$ such that $||u(t)||_{L^p} < \infty$, i.e. $\left(\int_{\mathbb{R}^n} (||u(t)||_Z)^p dt\right)^{\frac{1}{p}} < \infty.$

Example 1.71. *For $u \in \mathcal{S}(\mathbb{R}^n)$ and a nonnegative real number s we define* fractional powers of the Laplacian *by*

$$(-\triangle)^s u := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2s}\hat{u}(\xi)d\xi.$$

Note that when $s = 1$ this coincides with the usual definition of the Laplacian since $(-\triangle u)^\wedge(\xi) = |\xi|^2\hat{u}(\xi)$. Indeed,

$$\begin{aligned}(-\triangle u)^\wedge(\xi) &= -\left(\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}\right)^\wedge(\xi)\\ &= -\sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_j^2}\right)^\wedge(\xi)\end{aligned}$$

$$= -\sum_{j=1}^{n}(-i\xi_j)^2\hat{u}(\xi)$$

$$= -(-i)^2\sum_{j=1}^{n}(\xi_j)^2\hat{u}(\xi)$$

$$= \sum_{j=1}^{n}(\xi_j)^2\hat{u}(\xi)$$

$$= |\xi|^2\hat{u}(\xi).$$

The operator $(-\triangle)^s$ maps $H^{2s}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. To see this note that

$$||(-\triangle)^s u||^2_{L^2(\mathbb{R}^n)} = ||F((-\triangle)^s u)||^2_{L^2(\mathbb{R}^n)}$$

$$= |||\cdot|^{2s}\hat{u}(\cdot)||^2_{L^2(\mathbb{R}^n)}$$

$$\leq ||(1+|\cdot|^2)^s\hat{u}(\cdot)||^2_{L^2(\mathbb{R}^n)} = ||u||^2_{H^{2s}(\mathbb{R}^n)}$$

where the first inequality followed from Plancherel's theorem. In fact, for all $t \geq 0$, the operator $(-\triangle)^s$ maps $H^{2s+t}(\mathbb{R}^n)$ to $H^t(\mathbb{R}^n)$. The proof of this is as follows:

$$||(-\triangle)^s u||^2_{H^t(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|\xi|^2)^t||\xi|^{2s}\hat{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^n}(1+|\xi|^2)^t(|\xi|^2)^{2s}|\hat{u}(\xi)|^2 d\xi$$

$$\leq \int_{\mathbb{R}^n}(1+|\xi|^2)^{2s+t}|\hat{u}(\xi)|^2 d\xi$$

$$= ||u||^2_{H^{2s+t}(\mathbb{R}^n)}.$$

Consider the following initial-value problem:

$$\begin{cases} u_t + (-\triangle)^s u &= 0 \\ u(x,0) &= g(x), \end{cases}$$

where $g \in L^2(\mathbb{R}^n)$. Multiplying both sides of the equation by a test function $v \in C_0^\infty(\mathbb{R}^n)$, integrating with respect to x over $\mathbb{R}^n$ and integrating with respect to t over $[0,T]$, we have

$$\int_{\mathbb{R}^n}\int_{[0,T]} u_t v\ dtdx + \int_{\mathbb{R}^n}\int_{[0,T]}(-\triangle)^s u\cdot v\ dtdx = 0.$$

Integrating the first term by parts we obtain

$$-\int_{\mathbb{R}^n}\int_{[0,T]} uv_t\ dtdx + \int_{\mathbb{R}^n} u(x,T)v(x,T)dx$$
$$-\int_{\mathbb{R}^n} u(x,0)v(x,0)dx + \int_{\mathbb{R}^n}\int_{[0,T]}(-\triangle)^s u\cdot v\ dtdx = 0.$$

By definition of $(-\triangle)^s$ *we have*

$$\begin{aligned}\int_{[0,T]}\int_{\mathbb{R}^n}(-\triangle)^s u\cdot v\ dxdt &= \int_{[0,T]}\int_{\mathbb{R}^n} F^{-1}(|\xi|^{2s}\hat{u}(\xi))(x)v(x)dxdt\\ &= \int_{[0,T]}\int_{\mathbb{R}^n}(|\xi|^{2s}\hat{u}(\xi))\bar{\hat{v}}(\xi)d\xi dt\\ &= \int_{[0,T]}\int_{\mathbb{R}^n}|\xi|^s\hat{u}(\xi)|\xi|^s\bar{\hat{v}}(\xi)d\xi dt\\ &= \int_{[0,T]}\int_{\mathbb{R}^n} F^{-1}(|\cdot|^s\hat{u}(\cdot))(x)F^{-1}(|\cdot|^s\bar{\hat{v}}(\cdot))(x)dxdt\\ &= \int_{[0,T]}\int_{\mathbb{R}^n}(-\triangle)^{\frac{s}{2}}u\cdot(-\triangle)^{\frac{s}{2}}v\ d\xi dt\end{aligned}$$

where the second and fourth equalities follow from Plancherel's theorem. Therefore a weak formulation of the problem is

$$\int_{\mathbb{R}^n} u(x,T)v(x,T)dx - \int_{\mathbb{R}^n}\int_{[0,T]} uv_t\ dtdx$$
$$+\int_{[0,T]}\int_{\mathbb{R}^n}(-\triangle)^{\frac{s}{2}}u\cdot(-\triangle)^{\frac{s}{2}}v dxdt = \int_{\mathbb{R}^n} g(x)v(x,0)dx.$$

Taking $v = u$ *in the weak formulation we obtain*

$$\int_{\mathbb{R}^n} u^2(x,T))dx - \int_{\mathbb{R}^n}\int_{[0,T]} uu_t\ dtdx$$
$$+\int_{[0,T]}\int_{\mathbb{R}^n}|(-\triangle)^{\frac{s}{2}}u|^2 dxdt = \int_{\mathbb{R}^n} g^2(x)dx.$$

Since $uu_t = \frac{1}{2}(u^2)_t$ *and*

$$-\int_{\mathbb{R}^n}\int_{[0,T]}\frac{1}{2}(u^2)_t dtdx = -\frac{1}{2}\int_{\mathbb{R}^n} u^2(x,T)dx + \frac{1}{2}\int_{\mathbb{R}^n} g^2(x)dx$$

we have

$$\frac{1}{2}\int_{\mathbb{R}^n} u^2(x,T)dx + \int_{[0,T]}\int_{\mathbb{R}^n}|(-\triangle)^{\frac{s}{2}}u|^2 dxdt = \frac{1}{2}\int_{\mathbb{R}^n} g^2(x)dx.$$

The term $\frac{1}{2}\int_{\mathbb{R}^n} u^2(x,T)dx$ *is always nonnegative hence*

$$\int_{[0,T]}\int_{\mathbb{R}^n}|(-\triangle)^{\frac{s}{2}}u|^2 dxdt \le \frac{1}{2}\int_{\mathbb{R}^n} g^2(x)dx,$$

which yields $u \in L^2([0,T]; H^s(\mathbb{R}^n))$. *The term* $\int_{[0,T]}\int_{\mathbb{R}^n} |(-\triangle)^{\frac{s}{2}}u|^2 dxdt$ *is also always nonnegative, hence*

$$\frac{1}{2}\int_{\mathbb{R}^n} u^2(x,t)dx \leq \frac{1}{2}\int_{\mathbb{R}^n} g^2(x)dx$$

for $t \in [0,T]$. *Taking the essential supremum of both sides yields*

$$\operatorname{esssup}_{0<t\leq T}\frac{1}{2}\int_{\mathbb{R}^n} u^2(x,t)dx \leq \frac{1}{2}\int_{\mathbb{R}^n} g^2(x)dx.$$

Thus $u \in L^\infty([0,T]; L^2(\mathbb{R}^n))$. *Combining these two results for* u *we finally arrive at*

$$u \in L^2([0,T]; H^s(\mathbb{R}^n)) \cap L^\infty([0,T]; L^2(\mathbb{R}^n)).$$

Example 1.72. *We can replace* $(-\triangle)^s$ *in Example 1.70 with the operator* $\psi(D)$ *given on* $\mathcal{S}(\mathbb{R}^n)$ *by*

$$\psi(D)u := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi$$

where $\psi(\xi) : \mathbb{R}^n \to \mathbb{R}$ *is a continuous negative definite function. The initial-value problem*

$$\left\{ \begin{array}{rcl} u_t + \psi(D)u & = & 0 \\ u(x,0) & = & g(x) \end{array} \right.$$

where $g \in L^2(\mathbb{R}^n)$ *and* $\psi(D)$ *is as above, has a solution in the space*

$$L^2([0,T]; H^{\psi,1}(\mathbb{R}^n)) \cap L^\infty([0,T]; L^2(\mathbb{R}^n)).$$

Remark 1.73. *The operator* $(-\triangle)$ *is a special case of* $\psi(D)$ *with* $\psi(\xi) = |\xi|^2$. *Furthermore,* $\psi(D)$ *is itself part of a larger class of operators that will be discussed in the following section.*

1.6 Pseudo-differential Operators

This section focuses on particular types of operator known as pseudo-differential operators and investigates the conditions necessary for them to be generators of semigroups. (See [18]).

Definition 1.74. *Let* $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *be a measurable function such that* $\xi \mapsto q(x,\xi)$ *is continuous and polynomially bounded. The operator* $q(x,D)$ *given by*

$$(1.43)\quad q(x,D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)d\xi$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, *is called a* pseudo-differential operator *with* symbol $q(x,\xi)$.

Theorem 1.75. ([18], Thm 1.7). *Let* $A : C_0^\infty(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ *be a linear operator satisfying the positive maximum principle. Then the following hold:*

(i) *There exists a locally bounded, measurable function* $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *such that* $\xi \mapsto q(x,\xi)$ *is a continuous negative definite function with representation*

$$(1.44)\quad q(x,\xi) = a(x) - il(x)\cdot\xi + \xi\cdot\mathcal{Q}(x)\xi + \int_{y\neq 0}\left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2}\right)N(x,dy).$$

Here the coefficients $a : \mathbb{R}^n \to [0,\infty)$, $l : \mathbb{R}^n \to \mathbb{R}^n$ *are measurable, the matrix* $\mathcal{Q}$ *is positive semidefinite and* $N(x,dy)$ *is a Lévy kernel on* $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ *such that* $\int_{y\neq 0}\left(|y|^2 \wedge 1\right)N(x,dy) < \infty$ *for all* $x \in \mathbb{R}^n$.

(ii) $Au(x) = -q(x,D)u(x)$.

□

The above result is originally due to Ph. Courrège [4].

If a pseudo-differential operator $q(x,D)$ has symbol $q(x,\xi)$ such that

(1.45) $\xi \mapsto q(x,\xi)$ is a continuous negative definite function,

then $q(x,D)$ satisfies the positive maximum principle. Recall that the generator of a Feller semigroup satisfies the positive maximum principle. We are therefore interested in pseudo-differential opeators with property (1.45).

Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function such that

(1.46) $\psi(\xi) \geq c_0|\xi|^{2r_0}$, $|\xi|$ large,

for constants $c_0, r_0 > 0$. Suppose also that $q(x,D)$ is a pseudo-differential operator with symbol satisfying (1.45). Set, for some $x_0 \in \mathbb{R}^n$,

$$q(x,\xi) = q_1(\xi) + q_2(x,\xi) := q(x_0,\xi) + (q(x,\xi) - q(x_0,\xi))$$

and assume the following:

$(P1)$ $\gamma_0\psi(\xi) \leq Re\ q_1(\xi) \leq \gamma_1\psi(\xi)$, for scalars $\gamma_0, \gamma_1 > 0$ and $|\xi| \geq 1$;

$(P2)$ $|Im\ q_1(\xi)| \leq \gamma_2(Re\ q_1(\xi))$, $\xi \in \mathbb{R}^n$;

$(P3)$ $q_2(\cdot, \xi) \in C^{m_0}(\mathbb{R}^n)$, where $m_0 := \left[\frac{n}{r_0}\right] + n + 3$ such that
$|\partial_x^\alpha q_2(x,\xi)| \leq \phi_\alpha(x)(1+\psi(\xi)),\ \phi_\alpha \in L^1(\mathbb{R}^n),\ |\alpha| \leq m_0$;

$(P4)$ $\sum_{|\alpha|\leq m_0} ||\phi_\alpha||_{L^1} \leq c_{n,m_0,\psi}\gamma_0$, where $c_{n,m_0,\psi}$ is a known constant.

Then we have

Theorem 1.76. ([18], Thm 3.3). *Under assumptions $(P1) - (P4)$ the operator $(-q(x,D), H^{\psi,t_0+2}(\mathbb{R}^n))$, where $t_0 := \left[\frac{n}{r_0}\right] + 1$, extends to the generator of a Feller semigroup. Moreover, there exists some $\lambda_0 \geq 0$ such that for $\lambda \geq \lambda_0$ the operator $-q_\lambda(x,D) = -\lambda\mathrm{id} - q(x,D)$ generates an L^p-sub-Markovian semigroup for $2 \leq p < \infty$.* □

Denote by $S_\rho^{m,\psi}$ the class of all symbols $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which are arbitrarily often differentiable and satisfy

(1.47) $|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq c_0(1+\psi(\xi))^{(m-\rho(|\alpha|))/2},\ \rho(k) := k \wedge 2.$

The following is due to W. Hoh:

Theorem 1.77. [11]. *Let $q(x,\xi) \in S_\rho^{m,\psi}$ be a real-valued symbol satisfying*

(1.48) $q(x,\xi) \geq \gamma_0(1+\psi(\xi))^{\frac{m}{2}},\ |\xi|$ *large.*

Then

(1.49) $||q(x,D)u||_{\psi,s} \leq c_1||u||_{\psi,m+s}$ *and*

(1.50) $||u||_{\psi,m+s} \leq c_2(||q(x,D)u||_{\psi,s} + ||u||_{\psi,s+m-\frac{1}{2}}).$ □

We also have

Theorem 1.78. ([18], Thm 3.4). *Let ψ and $q \in S_\rho^{m,\psi}$ satisfy* (1.46) *and* (1.48) *respectively and assume $\xi \mapsto q(x,\xi)$ is a continuous negative definite function. Then $(-q(x,D), C_0^\infty(\mathbb{R}^n))$ extends to the generator of a Feller semigroup. Moreover, for some $\lambda_0 \geq 0$, the operator $-q_\lambda(x,D) = -\lambda\mathrm{id} - q(x,D)$, $\lambda \geq \lambda_0$, generates an L^p-sub-Markovian semigroup for $2 \leq p < \infty$.* □

The next theorem is again due to W. Hoh:

Theorem 1.79. ([10], Thm 3.9). *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying* (1.46) *and let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $\xi \mapsto q(x,\xi)$ is negative definite and $q(x,0) \equiv 0$. Assume also*

1. $q(\cdot,\xi) \in C^{2m_0+1-n}(\mathbb{R}^n)$;
2. $|\partial_x^\beta q(x,\xi)| \le c_\beta(1+\psi(\xi))$, $|\beta| \le 2m_0+1-n$, *where* $m_0 := \left(\left[\frac{2n}{r_0}\right] \vee 2\right) + n + 1$;
3. *there exists a function* $\gamma : \mathbb{R}^n \to (0,\infty)$ *such that* $q(x,\xi) \ge \gamma(x)\phi(\xi)$ *for* $|\xi|$ *large;*
4. *there exists a continuous function* $\rho : \mathbb{R}^n \to [0,\infty)$ *such that* $\rho(0) = 0$ *and* $q(x,\xi) \le \rho(\xi)$ *where* $|\xi| \le \epsilon$ *for some* $\epsilon > 0$.

Then $-q(x,D)$ maps $C_0^\infty(\mathbb{R}^n)$ into $C_\infty(\mathbb{R}^n)$ and extends to the generator of a Feller semigroup. □

1.7 The Galerkin Procedure

The Galerkin procedure, detailed below, is a useful method of ascertaining the existence of and obtaining solutions to partial differential equations. (See [27]).

Definition 1.80. *Let X be a Banach space. A* Galerkin scheme *in X is a sequence (Y_n) of finite-dimensional nonzero subspaces Y_n of X with*

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{u}, \mathrm{Y_n}) = 0$$

for all $u \in X$, where $\mathrm{dist}(\mathrm{u},\mathrm{Y_n}) = \inf_{\mathrm{v}\in\mathrm{Y_n}} \|\mathrm{u}-\mathrm{v}\|$.
A basis *of X is an at most countable sequence (w_j) of elements $w_j \in X$, where finitely many $w_1,\dots,w_n$ are always linearly independent and X is the closure of $\bigcup_n X_n$ with $X_n = \mathrm{span}\{w_1,\dots,w_n\}$.*

Theorem 1.81. ([27], §21.13, Proposition 21.49). *Let X be a separable Banach space. Then*

(a) X has a basis.

(b) If (w_n) is a basis in X, then (X_n) with $X_n = \mathrm{span}\{w_1,\dots,w_n\}$ is a Galerkin scheme in X.

(c) If (Y_n) is a Galerkin scheme in X, then we can construct a basis in X by means of (Y_n).

□

Let H and V be real Hilbert spaces. If $X = L^p(0,T;V)$ we have $X^* = L^q(0,T;V^*)$ where $p^{-1}+q^{-1}=1$ and X^* denotes the dual space of X. We now introduce the space

$$W_p^1(0,T:V,H) := \{u \in X : u' \in X^*\}, \quad 1 < p < \infty,$$

which will be a useful solution space. We also need

Definition 1.82. *We call $V \subseteq H \subseteq V^*$ an* evolution triple *if*

(i) V is a real, separable and reflexive Banach space;

(ii) H is a real, separable Hilbert space;

(iii) The embedding $V \subseteq H$ is continuous, i.e., $||v||_H \leq c||v||_V$ (where c is constant) for all $v \in V$, and V is dense in H.

Consider the following initial-value problem along with assumptions (H1), (H2), (H3) below:

$$(1.51) \qquad \begin{cases} \frac{d}{dt}(u(t)|v)_H + a(u(t),v) &= \; < b(t), v >_V \\ u(0) &= \; u_0 \in H \end{cases}$$

with $u \in W_2^1(0,T;V,H)$. Here $(\;|\;)_H$ denotes the inner product in H.

(H1) "$V \subseteq H \subseteq V^*$" is an evolution triple with $\dim V = \infty, 0 < T < \infty$. The spaces V and H are real Hilbert spaces.

(H2) The mapping $a : V \times V \to \mathbb{R}$ is bilinear, bounded and strongly positive. Moreover, we are given $u_0 \in H$ and $b \in L^2(0,T;V^*)$.

(H3) $\{w_1, w_2, \ldots\}$ is a basis in V, and (u_{n0}) is a sequence from H with $u_{n0} \to u_0$ in H as $n \to \infty$ where $u_{n0} \in \text{span}\{w_1, \ldots, w_n\}$ for all n.

We formulate the Galerkin method as follows:
Set

$$u_n(t) = \sum_{k=1}^{n} c_{kn}(t) w_k, \quad u_{n0} = \sum_{k=1}^{n} \alpha_{kn} w_k.$$

Definition 1.83. *For almost all $t \in (0,T)$ the* **Galerkin equations** *are*

$$(1.52) \qquad \begin{cases} \sum_{k=1}^{n} c_{kn}'(t)(w_k|w_j)_H + c_{kn}(t)a(w_k,w_j) &= \; < b(t), w_j >_V \\ c_{jn}(0) &= \; \alpha_{jn}. \end{cases}$$

Theorem 1.84. ([27], §23.7, Thm 23.A). *If assumptions (H1), (H2) and (H3) hold then*

(*i*) (1.51) *has exactly one solution u.*

(*ii*) *The map $(u_0, b) \mapsto u$ is linear and continuous from $H \times L^2(0,T;V^*)$ to $W_2^1(0,T;V,H)$.*

(*iii*) *For $n = 1, 2, \ldots$ the Galerkin equations* (1.52) *have exactly one solution $u_n \in W_2^1(0,T;V,H)$. The sequence (u_n) converges as $n \to \infty$ to the solution u of* (1.51) *in the following sense:*

$$u_n \to u \text{ in } L^2(0,T;V), \ \max_{0 \le t \le T} \|u_n(t) - u(t)\|_H \to 0.$$

□

Example 1.85. ([27], §23.8). *The Galerkin method can be applied to an initial-boundary value problem of the form*

$$(1.53) \qquad \begin{cases} u_t(x,t) - \Delta u(x,t) &= f(x,t) \text{ on } G \times (0,T) \\ u(x,t) &= 0 \text{ on } \partial G \times [0,T] \\ u(x,0) &= u_0(x) \text{ on } G, \end{cases}$$

where $G \subset \mathbb{R}^N$ is bounded and $f \in L^2(G \times (0,T))$ and v_0 are given. The weak formulation for (1.53) *is:*

Let $V = W_2^1(G), H = L^2(G)$. We are looking for $u \in W_2^1(0,T;V,H)$ with $u(0) = u_0$ such that

$$\frac{d}{dt}(u(t)|v)_H + a(u(t), v) =< b(t), v >_V$$

for all $v \in V$ and *almost all $t \in (0,T)$.* Here, *for all $u, v \in V$*

$$a(u,v) = \int_G \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx,$$

$$< b(t), v >_V = \int_G f(x,t) v(x) dx$$

and

$$(u|v)_H = \int_G u(x)v(x)dx.$$

Chapter 2

Approximating Solutions of Some First Order Equations

In Chapter 1 we saw that it is not always possible to solve problems for first order partial differential equations in the classical sense. This led to the introduction of weak solutions and the notion of solving a partial differential equation in the weak sense. We also found that under certain conditions the particular type of weak solutions discussed were unique. In this chapter we will consider the same first order equations but with additional second order terms that have a coefficient, ϵ, where $\epsilon > 0$ is a small constant. The aim is to show that the limit as $\epsilon \to 0$ of the solutions of such equations is in fact the solution of the corresponding equation (when $\epsilon = 0$) as found in Chapter 1. In Chapter 3 we will vary the choice of this extra term in the hope of arriving at the same result. This would provide yet another method of finding weak solutions and it would give some further, purely mathematical, justification of current efforts in modelling with Burgers-type equations.

We begin with the one-dimensional initial-value problem

$$(2.1) \qquad \left\{ \begin{array}{rcl} u_t(x,t) + \eta u_x(x,t) - \epsilon u_{xx}(x,t) & = & 0 \\ u(x,0) & = & g(x), \end{array} \right.$$

where $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is to be found, $g \in \mathcal{S}(\mathbb{R})$ (or $g \in C_\infty(\mathbb{R}) \cap L^1(\mathbb{R})$), $\eta \in \mathbb{R}$ and $\epsilon > 0$. Taking the Fourier transform with respect to x in (2.1) we get

$$\left\{ \begin{array}{rcl} \hat{u}_t(\xi,t) + \eta i \xi \hat{u}(\xi,t) + \epsilon \xi^2 \hat{u}(\xi,t) & = & 0 \\ \hat{u}(\xi,0) & = & \hat{g}(\xi) \end{array} \right.$$

or

$$\begin{cases} \hat{u}_t(\xi,t) &= -(\epsilon\xi^2 + i\eta\xi)\hat{u}(\xi,t) \\ \hat{u}(\xi,0) &= \hat{g}(\xi). \end{cases}$$

This initial-value problem for a linear ordinary differential equation with constant coefficients has the unique solution

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-(\epsilon\xi^2+i\eta\xi)t}$$

which yields

$$\begin{aligned} u(x,t) &= F^{-1}(\hat{g}(\xi)e^{-(\epsilon\xi^2+i\eta\xi)t}) \\ &= (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{ix\xi}\hat{g}(\xi)e^{-(\epsilon\xi^2+i\eta\xi)t}d\xi \\ &= (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{i\xi(x-\eta t)}e^{-\epsilon\xi^2 t}\hat{g}(\xi)d\xi. \end{aligned}$$

Putting $z = x - \eta t$ we get

$$u^\epsilon(x,t) = (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{i\xi z}e^{-\epsilon\xi^2 t}\hat{g}(\xi)d\xi$$

as a solution of (2.1). However, we can rewrite this as follows:

$$u^\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\int_{\mathbb{R}} e^{-\frac{|z-y|^2}{4\epsilon t}}g(y)dy = (T_{\epsilon t}g)(z) = (T_{\epsilon t}g)(x-\eta t),$$

where $(T_t)_{t\geq 0}$ is the one-dimensional Gaussian semigroup. Using standard continuity arguments it can be shown that the last line holds for all $g \in C_\infty(\mathbb{R}) \cap C^1(\mathbb{R})$. By the strong continuity of the Gaussian semigroup, for $g \in C_\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ we get

$$\lim_{\epsilon\to 0}\|u^\epsilon(\cdot,t) - g(\cdot - \eta t)\|_\infty = 0.$$

In particular $u^\epsilon(x,t) \to g(x-\eta t)$ as $\epsilon \to 0$. However,

$$\frac{\partial}{\partial t}g(x-\eta t) + \eta\frac{\partial}{\partial x}g(x-\eta t) = -\eta g'(x-\eta t) + \eta g'(x-\eta t) = 0.$$

Thus $v(x,t) := g(x-\eta t)$ solves

$$\begin{cases} v_t(x,t) + \eta v_x(x,t) &= 0 \\ v(x,0) &= g(x). \end{cases}$$

Therefore as $\epsilon \to 0$ the solution $u^\epsilon(x,t) = (T_{\epsilon t}g)(x-\eta t)$ of

$$(2.1) \qquad \begin{cases} u_t(x,t) + \eta u_x(x,t) - \epsilon u_{xx}(x,t) &= 0 \\ u(x,0) &= g(x) \end{cases}$$

tends in the norm $\|\cdot\|_\infty$ to $v(x,t) = g(x - \eta t)$ which is a solution of

$$\begin{cases} v_t(x,t) + \eta v_x(x,t) &= 0 \\ v(x,0) &= g(x). \end{cases}$$

Now let us consider, in n-dimensions, the more general equation

$$\begin{cases} u_t(x,t) + \eta b \cdot \nabla u(x,t) + \epsilon\psi(D_x)u(x,t) &= 0 \\ u(x,0) &= g(x) \end{cases} \tag{2.2}$$

where $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is to be found, $g \in \mathcal{S}(\mathbb{R}^n)$ (or $g \in C_\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$), $\eta \in \mathbb{R}$, $b \in \mathbb{R}^n, \epsilon > 0$ and the pseudo-differential operator $-\psi(D)$ is the generator of a Feller semigroup $(T_t)_{t\geq 0}$. More precisely, we assume that $\psi : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$) is a continuous negative definite function associated with the convolution semigroup $(\mu_t)_{t\geq 0}$ by $\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$. The semigroup of operators associated with $(\mu_t)_{t\geq 0}$ will be denoted by $(T_t^\psi)_{t\geq 0}$ or simply by $(T_t)_{t\geq 0}$ if there is no possibility of confusion. As we have seen before (see Chapter 1, Example 1.59), the generator A of $(T_t^\psi)_{t\geq 0}$ has, on $\mathcal{S}(\mathbb{R}^n)$, a representation as a pseudo-differential operator with symbol $-\psi$. Indeed, for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} Au(x) &= \lim_{t\to 0} \frac{T_t^\psi u(x) - u(x)}{t} \\ &= -\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi. \end{aligned} \tag{2.3}$$

Returning to (2.2) and taking the Fourier transform with respect to x we get

$$\begin{cases} \hat{u}_t(\xi,t) + i\eta b\cdot\xi\hat{u}(\xi,t) + \epsilon\psi(\xi)\hat{u}(\xi,t) &= 0 \\ \hat{u}(\xi,0) &= \hat{g}(\xi) \end{cases}$$

or

$$\begin{cases} \hat{u}_t(\xi,t) &= -(\epsilon\psi(\xi) + i\eta b\cdot\xi)\hat{u}(\xi,t) \\ \hat{u}(\xi,0) &= \hat{g}(\xi) \end{cases}$$

which has the unique solution $\hat{u}(\xi,t) = \hat{g}(\xi)e^{-(\epsilon\psi(\xi)+i\eta b.\xi)t}$. Hence, for $u \in \mathcal{S}(\mathbb{R}^n)$

$$u^\epsilon(x,t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{g}(\xi)e^{-(\epsilon\psi(\xi)+i\eta b.\xi)t}d\xi$$

solves (2.2). Putting $z = x - \eta t b$ we get

$$u^\epsilon(x,t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot z}e^{-\epsilon\psi(\xi)t}\hat{g}(\xi)d\xi = (T_{\epsilon t}g)(z). \tag{2.4}$$

Note that the right-hand side of (2.4) is well-defined for all $g \in C_\infty(\mathbb{R}^n)$ because $(T_t)_{t\geq 0}$ is a Feller semigroup. From the strong continuity of $(T_t)_{t\geq 0}$ we get, for $g \in \mathcal{S}(\mathbb{R}^n)$ (or g in $C_\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ or $C_\infty(\mathbb{R}^n)$),

$$\lim_{\epsilon\to 0} \|u^\epsilon(\cdot - \eta t b) - g(\cdot - \eta t b)\|_\infty = 0,$$

or in particular $u^\epsilon(x,t) \to g(x - \eta t b)$ as $\epsilon \to 0$. However

$$\frac{\partial}{\partial t} g(x - \eta t b) + \eta b \cdot \nabla_x g(x - \eta t b) = -\eta b \cdot \nabla_{z=x-\eta t b} g + \eta b \cdot \nabla_{z=x-\eta t b} g = 0.$$

Hence $v(x,t) := g(x - \eta t b)$ solves

$$(2.5)\quad \begin{cases} v_t(x,t) + \eta b \cdot \nabla_x v(x,t) &= 0 \\ v(x,0) &= g(x). \end{cases}$$

Therefore, for $g \in \mathcal{S}(\mathbb{R}^n)$, as $\epsilon \to 0$ the solution $u^\epsilon(x,t) = (T_{\epsilon t} g)(x - \eta t b)$ of

$$(2.2)\qquad \begin{cases} u_t(x,t) + \eta b \cdot \nabla u(x,t) + \epsilon \psi(D_x) u(x,t) &= 0 \\ u(x,0) &= g(x) \end{cases}$$

tends to the solution $v(x,t)$ of (2.5). We need to mention a certain regularity problem. Although $u^\epsilon(x,t) = (T_{\epsilon t} g)(x - \eta t b)$ is well-defined for all $g \in C_\infty(\mathbb{R}^n)$ and $\lim_{\epsilon\to 0} ||u^\epsilon - g||_\infty = 0$, we cannot claim a priori that $u^\epsilon(x,t)$, $g \in C_\infty(\mathbb{R}^n)$, solves (2.2) since our construction uses the Fourier transform of g and the fact that $F^{-1}(\hat{g} e^{-(\epsilon\psi + i\eta b\cdot\xi)t})$ defines a smooth function. This situation could be remedied by assuming that $\psi(\xi)$ has a certain decay as $|\xi| \to \infty$ or equivalently that

$$T_t u = \int_{\mathbb{R}^n} u(y) \rho(x - y, t) dy$$

where $\rho(\cdot, t)$ is a smooth function. Under such additional assumptions we will next consider a more general case.

Let $(S_t)_{t\geq 0}$ be a general Feller semigroup on $C_\infty(\mathbb{R}^n)$ with smooth density ρ and generator A and assume that $C_b^2(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$ is dense in $D(A)$. Now set $u^\epsilon(x,t) := (S_{\epsilon t} g)(x - \eta t b)$ where $g \in C_b^2(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$, $b \in \mathbb{R}^n$ and $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$. Then

$$(2.6)\qquad \begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \epsilon A u^\epsilon - \eta b \cdot \nabla u^\epsilon &= 0 \\ u(x,0) &= g(x) \end{cases}$$

holds. Indeed setting $X_j := x_j - \eta t b_j$, $Y_j := y_j$, and $Z := \epsilon t$ we have via the chain rule,

$$\frac{\partial}{\partial t}(S_{\epsilon t}g)(x-\eta tb) = \frac{\partial}{\partial t}\int_{\mathbb{R}^n}\rho(\epsilon t, x-\eta tb, y)g(y)dy$$

$$= \epsilon\frac{\partial}{\partial Z}\int_{\mathbb{R}^n}\rho(Z,X,Y)g(Y)dY - \eta\sum_{j=1}^{n} b_j\frac{\partial}{\partial X_j}\int_{\mathbb{R}^n}\rho(Z,X,Y)g(Y)dY$$

$$= \epsilon\frac{\partial}{\partial Z}(S_Z g)(X) - \eta\sum_{j=1}^{n} b_j\frac{\partial}{\partial X_j}(S_Z g)(X)$$

$$= \epsilon A(S_{\epsilon t}g)(x-\eta tb) - \eta\sum_{j=1}^{n} b_j\frac{\partial}{\partial x_j}(S_{\epsilon t}g)(x-\eta tb),$$

where the last equality follows from (1.41), Chapter 1. Let us study the limit as $\epsilon \to 0$ of this equation. Since $g \in C_b^2(\mathbb{R}^n)\cap C_\infty(\mathbb{R}^n) \subset D(A)$ it follows that $S_{\epsilon t}g \in D(A)$ and $AS_{\epsilon t}g = S_{\epsilon t}Ag$, hence using the contraction property of $S_{\epsilon t}$ we find $||AS_{\epsilon t}g||_\infty \le ||Ag||_\infty$. Therefore we may conclude that $||\epsilon AS_{\epsilon t}g||_\infty \to 0$ as $\epsilon \to 0$. Furthermore, by the argument used in the previous cases we find that $S_{\epsilon t}g \to g$ in $C_\infty(\mathbb{R}^n)$, so in the limit as $\epsilon \to 0$ equation (2.6) becomes

$$\text{(2.7)} \qquad \left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} + \eta b\cdot\nabla u &=& 0\\ u(x,0) &=& g(x), \end{array}\right.$$

where $u(x,t) := g(x-\eta tb) = \lim_{\epsilon\to 0}(S_{\epsilon t}g)(x-\eta tb)$. Hence $\lim_{\epsilon\to 0} u^\epsilon(x,t) = u(x,t)$. These results are summarized as follows:

Theorem 2.1. *Let $(S_t)_{t\ge 0}$ be a Feller semigroup on $C_\infty(\mathbb{R}^n)$ with smooth density. Suppose that $C_b^2(\mathbb{R}^n)\cap C_\infty(\mathbb{R}^n) \subset D(A)$ is dense and define for $g \in C_b^2(\mathbb{R}^n)\cap C_\infty(\mathbb{R}^n)$ the function $u^\epsilon(x,t) := (S_{\epsilon t}g)(x-\eta tb)$. Then u^ϵ solves* (2.6) *and as $\epsilon \to 0$ the function tends to a solution of*

$$\text{(2.7)} \qquad \left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} + \eta b\cdot\nabla u &=& 0\\ u(x,0) &=& g(x). \end{array}\right.$$

Although we know of no reference for Theorem 2.1, it is not a surprising result bearing in mind standard properties of semigroups.

Our interest in such a result grew out of considerations related to the viscous Burgers equation. Let us now look at the initial-value problem for the viscous Burgers equation given by

$$(2.8) \qquad \begin{cases} u_t - \epsilon u_{xx} + uu_x &= 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u &= g \text{ on } \mathbb{R} \times \{t=0\}, \end{cases}$$

($\epsilon > 0$) where g is a smooth function. (See [7] §4.4.1). We first need to obtain a solution to the following quasilinear parabolic equation

$$(2.9) \qquad \begin{cases} u_t - \epsilon\triangle u + \eta|\nabla u|^2 &= 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

($\epsilon > 0$). Assume for the moment that u is a smooth solution of (2.9) and set $w := \phi(u)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function. We have

(*i*) $w_t = \phi'(u)u_t$

and

(*ii*) $\triangle w = \phi'(u)\triangle u + \phi''(u)|\nabla u|^2$.

For the proof of (*ii*) note first that $w_{x_i} = \phi'(u)u_{x_i}$ and $w_{x_ix_i} = \phi''(u)u_{x_i}u_{x_i} + \phi'(u)u_{x_ix_i}$. Then

$$\begin{aligned} \triangle w = \sum_{i=1}^n w_{x_ix_i} &= \phi''(u)\sum_{i=1}^n u_{x_i}^2 + \phi'(u)\sum_{i=1}^n u_{x_ix_i} \\ &= \phi''(u)|\nabla u|^2 + \phi'(u)\nabla u. \end{aligned}$$

Rearranging the partial differential equation in (2.9) to get $u_t = \epsilon\triangle u - \eta|\nabla u|^2$ and substituting this into (*i*) gives

$$\begin{aligned} w_t &= \phi'(u)(\epsilon\triangle u - \eta|\nabla u|^2) \\ &= \epsilon\triangle w - \epsilon\phi''(u)|\nabla u|^2 - \eta\phi'(u)|\nabla u|^2 \\ &= \epsilon\triangle w - (\epsilon\phi''(u) + \eta\phi'(u))|\nabla u|^2 \end{aligned}$$

where the second line follows from (*ii*). If we choose ϕ to satisfy $\epsilon\phi'' + \phi' = 0$ then we have

$$w_t = \epsilon\triangle w.$$

To solve this second order ordinary differential equation for ϕ note that the auxiliary equation $\epsilon k^2 + \eta k = 0$ has solutions $k = 0$ and $k = -\frac{\eta}{\epsilon}$ giving $\phi(z) = Ae^{-\frac{\eta z}{\epsilon}} + B$, $A, B \in \mathbb{R}$, as the general solution. Thus if we choose the particular solution $\phi(z) = e^{-\frac{\eta z}{\epsilon}}$ then

(2.10) $w = e^{-\frac{\eta u}{\epsilon}}$ (the Cole-Hopf transform)

solves

$$\text{(2.11)} \qquad \begin{cases} w_t - \epsilon \Delta w &= 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ w &= e^{-\frac{\eta g}{\epsilon}} \text{ on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where u is a solution of (2.9). The initial-value problem (2.11) for the heat equation with conductivity ϵ has the unique bounded solution

$$w(x,t) = \frac{1}{(4\pi\epsilon t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4\epsilon t}} e^{-\frac{\eta}{\epsilon} g(y)} dy$$

($x \in \mathbb{R}^n, t > 0$). Formula (2.10) yields $u = -\frac{\epsilon}{\eta} \ln w$, hence

$$u(x,t) = -\frac{\epsilon}{\eta} \ln \left(\frac{1}{(4\pi\epsilon t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{\eta g(y)}{\epsilon}} dy \right)$$

($x \in \mathbb{R}^n, t > 0$), solves (2.9).

Returning to (2.8), set

$$v(x,t) := \int_{-\infty}^{x} u(y,t) dy$$

and

$$h(x) := \int_{-\infty}^{x} g(y) dy.$$

Then we have

$$\begin{cases} v_t - \epsilon v_{xx} + \frac{1}{2} {v_x}^2 &= 0 \text{ in } \mathbb{R} \times (0,\infty) \\ v &= h \text{ on } \mathbb{R} \times \{t=0\}, \end{cases}$$

which is of the form (2.9) with $n = 1, \eta = \frac{1}{2}$. Thus

$$v(x,t) = -2\epsilon \ln \left(\frac{1}{(4\pi\epsilon t)^{\frac{1}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy \right).$$

Thus, since $u = v_x$, a solution of (2.8) is given by

$$\text{(2.12)} \qquad u^{\epsilon}(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy}{\displaystyle\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy}$$

($x \in \mathbb{R}, t > 0$).

We now investigate this solution as $\epsilon \to 0$.

Lemma 2.2. [7, § 4.5.2, p.205] *Suppose that $k, l : \mathbb{R} \to \mathbb{R}$ are continuous functions, that l grows at most linearly and that k grows at least quadratically. Assume also that there exists a unique point $y_0 \in \mathbb{R}$ such that*

$$k(y_0) = \min_{y \in \mathbb{R}} k(y).$$

Then

$$\lim_{\epsilon \to 0} \frac{\int_{-\infty}^{\infty} l(y) e^{-\frac{k(y)}{\epsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\epsilon}} dy} = l(y_0).$$

□

Rewrite (2.12) as

$$u^\epsilon(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{K(x,y,t)}{2\epsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{K(x,y,t)}{2\epsilon}} dy}$$

where $K(x,y,t) := \dfrac{|x-y|^2}{2t} + h(y)$, $(x, y \in \mathbb{R}, t > 0)$ and $h(y) := \int_{-\infty}^{x} g(y)dy$. It turns out that

$$K(x,y,t) = tL\left(\frac{x-y}{t}\right) + h(y)$$

where L is the Legendre transform of F (i.e.
$L(p) = F^*(p) := \sup_{q \in \mathbb{R}}\{p \cdot q - F(q)\} \quad (p \in \mathbb{R})$) and $F(z) = \dfrac{z^2}{2}$. Then from Lemma 2.2 it follows that

$$\lim_{\epsilon \to 0} u^\epsilon(x,t) = \frac{x - \tilde{y}(x,t)}{t}$$

which is the solution obtained in Chapter 1, §1.2 for (1.17). Therefore, as $\epsilon \to 0$ the solution

$$u^\epsilon(x,t) = \frac{\int_{\infty}^{\infty} \frac{x-y}{t} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy}{\int_{\infty}^{\infty} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy}$$

of (2.8) tends to the solution $u(x,t) = \dfrac{x - \tilde{y}(x,t)}{t}$ of (1.17).

The original idea for this thesis was to extend the above result by replacing the operator $-\triangle$ in Burgers' equation with a pseudo-differential operator $\psi(D)$. Work has been done on such equations by Biler, Karch and Woyczyński in [2] and others (see the introduction of this thesis). The desired result has not been attained. It seems that the case involving $-\triangle$ has much dependence on particular properties of the Laplacian, the heat equation and its solution. However, let us note the following probabilistic interpretation of formula (2.12):

Let $(Y_t)_{t\geq 0}$ be a 1-dimensional Brownian Motion. Then the corresponding semigroup $(T_t)_{t\geq 0}$ is given by

$$T_t g(x) = E^x(g(Y_t)) = \int_{\mathbb{R}} g(y) P^x_{Y_t}(dy)$$

where $P^x_{Y_t}$ denotes the distribution of Y_t under P^x, the probability measure corresponding to Brownian Motion starting at $x \in \mathbb{R}$. For Brownian Motion we know $P^x_{Y_t}$ explicitly, namely

$$P^x_{Y_t} = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x-y|^2}{4t}} \lambda^{(1)}$$

where $\lambda^{(1)}$ denotes the Lebesgue measure on $\mathbb{R}$. We may therefore rewrite the numerator of (2.12) as follows:

$$\begin{aligned}
&\int_{\infty}^{\infty} \frac{x-y}{t} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy \\
&\qquad = (4\pi\epsilon t)^{\frac{1}{2}} \int_{\infty}^{\infty} \frac{x-y}{t} e^{-\frac{h(y)}{2\epsilon}} P^x_{Y_t}(dy) \\
&\qquad = (4\pi\epsilon t)^{\frac{1}{2}} E^x\left(\frac{x-Y_t}{t} e^{-\frac{h(Y_t)}{2\epsilon}}\right).
\end{aligned}$$

Similarly the denominator of (2.12) can be rewritten as

$$\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4\epsilon t} - \frac{h(y)}{2\epsilon}} dy = (4\pi\epsilon t)^{\frac{1}{2}} E^x\left(e^{-\frac{h(Y_t)}{2\epsilon}}\right).$$

We then have from (2.12)

$$u^\epsilon(x,t) = \frac{E^x\left(\frac{x-Y_t}{t} e^{-\frac{h(Y_t)}{2\epsilon}}\right)}{E^x\left(e^{-\frac{h(Y_t)}{2\epsilon}}\right)} = \frac{E^x\left(\frac{x-Y_t}{t} e^{-\frac{1}{2\epsilon}\int_{-\infty}^{Y_t} g(y)dy}\right)}{E^x\left(e^{-\frac{1}{2\epsilon}\int_{-\infty}^{Y_t} g(y)dy}\right)}. \tag{2.13}$$

It would be of interest to know whether probabilistic techniques would make it possible to pass in (2.13) to the limit as $\epsilon \to 0$ for more general Feller processes $(Y_t)_{t\geq 0}$, or even only for certain Lévy processes.

Chapter 3

The Nonlinear Case

In this chapter we study pseudo-differential equations similar to those studied in Chapter 2 where ∇u is replaced by a term involving $f(u)$ for a smooth function f.

We begin with a result from [3] for the equation

$$(3.1) \quad \begin{cases} u_t + \mathfrak{L}u + \nabla \cdot f(u) &= 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) &= u_0(x) \text{ on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $-\mathfrak{L}$ is the generator of a translation invariant sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^1(\mathbb{R}^n)$ and $f : \mathbb{R} \to \mathbb{R}^n$, along with its first derivative, is a continuous function. Let the symbol, a, of $\mathfrak{L}$ have representation

$$(3.2) \quad a(\xi) = ib\xi + q(\xi) + \int_{\mathbb{R}^n} (1 - e^{-i\eta\xi} - i\eta\xi \mathbf{1}_{\{|\eta|<1\}}(\eta))\Pi(d\eta)$$

where $b \in \mathbb{R}$, $q(\xi)$ is a positive-definite quadratic form on $\mathbb{R}^n$ and Π is a Borel measure such that $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}^n} \min(1, |\eta|^2)\Pi(d\eta) < \infty$.

Definition 3.1. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $\mathfrak{X} = L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. A* mild solution *of* (3.1) *is a weakly continuous function $u \in C([0,T]; \mathfrak{X})$ satisfying*

$$(3.3) \quad u(t) = T_t u_0 - \int_0^t T_{t-s} \nabla \cdot f(u(s))ds$$

for each $t \in (0,T)$.

Formula (3.3) is called the *Duhamel formula* for (3.1) and its derivation is as follows: We want to solve

$$(A) \quad \begin{cases} u_t(x,t) + \mathfrak{L}u(x,t) &= -\nabla \cdot f(u(x,t)) \\ u(x,0) &= u_0(x). \end{cases}$$

We do this by adding solutions of

$$(B) \quad \begin{cases} u_t(x,t) + \mathfrak{L}u(x,t) &= -\nabla \cdot f(u(x,t)) \\ u(x,0) &= 0 \end{cases}$$

to solutions of

$$(C) \quad \begin{cases} u_t(x,t) + \mathfrak{L}u(x,t) &= 0 \\ u(x,0) &= u_0(x). \end{cases}$$

A solution of (C) is given by $u^C(x,t) = T_t u_0(x)$. To find a solution of (B) first note that $\tilde{u}^B(x,t) = T_t(-\nabla \cdot f(u(x,0)))$ solves

$$\begin{cases} u_t(x,t) + \mathfrak{L}u(x,t) &= 0 & t > 0 \\ u(x,0) &= -\nabla \cdot f(u(x,0)) & t = 0 \end{cases}$$

and hence a solution of

$$\begin{cases} u_t(x,t) + \mathfrak{L}u(x,t) &= 0 & t > s \\ u(x,s) &= -\nabla \cdot f(u(x,s)) & t = s \end{cases}$$

is given by $\tilde{u}^B(x,t;s) = T_{t-s}(-\nabla \cdot f(u(x,s)))$. Then by the Duhamel principle a solution of (B) is

$$u^B(x,t) = \int_0^t T_{t-s}(-\nabla \cdot f(u(x,s)))ds$$

So we have a solution of (3.1) given by

$$\begin{aligned} u(x,t) &:= u^B(x,t) + u^C(x,t) \\ &= T_t u_0(x) - \int_0^t \nabla \cdot T_{t-s} f(u(x,s))ds. \end{aligned}$$

The following theorem is from [3] (Thm 3.1).

Theorem 3.2. *Assume that $f \in C^1(\mathbb{R};\mathbb{R}^n)$ and $\mathfrak{L}$ has symbol given by* (3.2) *satisfying*

$$(3.4) \quad \limsup_{|\xi|\to\infty} \frac{a(\xi) - a_0|\xi|^2}{|\xi|^{\tilde{\alpha}}} < \infty.$$

For $u_0 \in \mathfrak{X} = L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, there exists a unique mild solution $u \in C([0,\infty); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of the problem (3.1).

Sketch of Proof: To prove the local existence result we show that the nonlinear operator

$$\mathfrak{N}(u)(t) = T_t u_0 - \int_0^t \nabla \cdot T_{t-\tau} f(u(\tau))d\tau$$

has a unique fixed point in the Banach space

$$\mathcal{X}_T = L^\infty((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$$

equipped with the norm

$$||u||_{\mathcal{X}_T} = \sup_{0<t<T} ||u(t)||_{L^1(\mathbb{R}^n)} + \sup_{0<t<T} ||u(t)||_{L^\infty(\mathbb{R}^n)}.$$

Thus a local-in-time mild solution of (3.1) is obtained, via the Banach contraction theorem, as a fixed point of $\mathcal{N}$ in the ball

$$B(u_0, R) \equiv \{u \in \mathcal{X}_T : ||u - u_0||_{\mathcal{X}_T} \le R\},$$

for sufficiently large $R \ge 3||u_0||_{\mathcal{X}_T}$ and small $T > 0$. This is an immediate consquence of the inequalities which hold for all $u, v \in B(u_0, R)$:

(3.5) $||\mathcal{N}(u)||_{\mathcal{X}_T} \le ||u_0||_{\mathcal{X}} + c_1||u||_{\mathcal{X}_T}$

and

(3.6) $||\mathcal{N}(u) - \mathcal{N}(v)||_{\mathcal{X}_T} \le c_2||u - v||_{\mathcal{X}_T}$

where c_1 and c_2 depend on T and R. (The proofs of (3.5) and (3.6) follow similar arguments to those of Lemmas 3.8 and 3.9 that appear later in this chapter and so we do not give the details here). Indeed, for small enough T we have $c_2 < 1$ and then by (3.6) $\mathcal{N}$ is a contraction mapping. We also have

$$\begin{aligned}
||\mathcal{N}(u) - u_0||_{\mathcal{X}_T} &\le ||\mathcal{N}(u)||_{\mathcal{X}_T} + ||u_0||_{\mathcal{X}_T} \\
&\le ||u_0||_{\mathcal{X}} + c_1||u||_{\mathcal{X}_T} + ||u_0||_{\mathcal{X}_T} \\
&\le 2||u_0||_{\mathcal{X}} + c_1||u||_{\mathcal{X}_T} \\
&\le \frac{2}{3}R + c_1||u||_{\mathcal{X}_T} \\
&< R
\end{aligned}$$

where T is small enough to ensure $c_1 < \dfrac{R}{3||u||_{\mathcal{X}_T}}$. Therefore $B(u_0, R)$ is invariant under $\mathcal{N}$. Hence the conditions for the Banach contraction theorem are satisfied if T is small enough such that $c_1 < \dfrac{R}{3||u||_{\mathcal{X}_T}}$ and $c_2 < 1$. □

At the end of Chapter 2 we considered the initial-value problem for the viscous Burgers equation

$$(B.E.)\qquad \begin{cases} u_t - \epsilon u_{xx} + uu_x &= 0 \text{ in } \mathbb{R}\times(0,\infty) \\ u &= g \text{ on } \mathbb{R}\times\{t=0\}, \end{cases}$$

and we discussed the limiting behaviour of its solution, u^ϵ, as $\epsilon \to 0$. In the above result from [3], the term ϵu_{xx} in $(B.E)$ was substituted by $\mathcal{L}$, the generator of a certain type of Lévy operator semigroup. Since $\mathcal{L}$ is translation invariant the commutator $[\mathcal{L}, \frac{\partial}{\partial x_j}]$ vanishes. We now consider a nonlinear problem, analogous to (3.1), involving a non translation invariant generator of an L^2-sub-Markovian semigroup. First we need the following definition:

Definition 3.3. *A strongly continuous semigroup $(T_t)_{t\geq 0}$ is called an* analytic *semigroup of angle θ, if the mapping $t \mapsto T_t$ has an analytic extension to the sector $S := \{z \in \mathbb{C}\setminus\{0\} : |\text{arg}z| < \theta - \frac{\pi}{2}\}$.*

Let $-q(x,D)$ be a pseudo-differential operator defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$-q(x,D)u(x) = -(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)d\xi$$

where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous function that is negative definite in ξ. (Note that now our pseudo-differential operator also has x-dependence). Suppose also that $-q(x,D)$ extends to the generator of a symmetric L^2-sub-Markovian semigroup $(T_t)_{t\geq 0}$. We again denote this extension by $-q(x,D)$. From [24] we know that $(T_t)_{t\geq 0}$ is analytic and that $-q(x,D)$ is a self-adjoint negative operator. Therefore the operator $[q(x,D)]^{\frac{1}{2}}$ is well-defined and we have

(3.7) $[[q(x,D)]^{\frac{1}{2}}, T_t] = 0$

where $[A,B] = AB - BA$ denotes the commutator of the operators A and B. From [22] (Chapter 2, Thm 6.13 C), using standard arguments, we have the estimate

(3.8) $||[q(x,D)]^{\frac{1}{2}}T_t|| \leq c_0 t^{-\frac{1}{2}}$,

for constant $c_0 > 0$. Noting that in terms of L^2-estimates we have $||(-\triangle)^{\frac{1}{2}}u||_{L^2} \sim ||\nabla u||_{L^2}$, it seems sensible to study the problem

$$(3.9)\qquad \begin{cases} u_t + q(x,D)u + [q(x,D)]^{\frac{1}{2}}f(u) &= 0 \text{ in } \mathbb{R}^n\times(0,\infty) \\ u(x,0) &= g(x) \text{ on } \mathbb{R}^n\times\{t=0\}. \end{cases}$$

Here $g \in L^2(\mathbb{R}^n)$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(0) = 0$ with bounded continuous first derivative (i.e. f has at most linear growth). We need to introduce the space

$$\mathcal{X}_T^{(2)} := L^\infty((0,T); L^2(\mathbb{R}^n))$$

with norm

$$||v||_{\mathcal{X}_T^{(2)}} := \sup_{0<t<T} ||v(t)||_{L^2}.$$

Definition 3.4. *A function* $u \in \mathcal{X}_T^{(2)}$ *is called a* mild solution *of* (3.9) *in the interval* $(0,T)$ *if*

$$(3.10)\quad u(x,t) = T_t g(x) - \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(u(x,s))ds$$

holds for all $0 < t < T$.

Remark 3.5. *If* $u \in D(q(x,D))$ *is a mild solution, then* u *solves* (3.9) *in* $L^2((0,T)\times\mathbb{R}^n)$ *(i.e.* u_t*,* $q(x,D)u$ *and* $[q(x,D)]^{\frac{1}{2}}u$ *belong to* $L^2((0,T)\times\mathbb{R}^n)$ *and equalities* (3.9) *hold).*

We will show that for any given $g \in L^2(\mathbb{R}^n)$ there exists $T_0 > 0$ such that (3.9) has unique mild solution in the interval $(0,T_0)$ by applying the Banach fixed point theorem to the operator $\mathcal{N} : \mathcal{X}_T^{(2)} \to \mathcal{X}_T^{(2)}$ defined by

$$(3.11)\quad \mathcal{N}(u)(x,t) := T_t g(x) - \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(u)(x,s)ds.$$

Remark 3.6. *Note that a fixed point of* $\mathcal{N}(u)$ *is in fact a mild solution of* (3.9) *due to* (3.7).

We need the following auxiliary results:

Lemma 3.7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function with a bounded and continuous derivative* f' *and suppose that* $f(0) = 0$*. For* $u \in \mathcal{X}_T^{(2)}$ *we have*

$$(3.12)\quad ||f(u)||_{\mathcal{X}_T^{(2)}} \le ||f'||_\infty ||u||_{\mathcal{X}_T^{(2)}}.$$

Proof. From the definition we have

$$\begin{aligned}
||f(u)||_{\mathcal{X}_T^{(2)}} &= \sup_{0<s<T} ||f(u(x,s))||_{L^2} \\
&= \sup_{0<s<T} \left(\int_{\mathbb{R}^n} |f(u(x,s))|^2 dx \right)^{\frac{1}{2}} \\
&= \sup_{0<s<T} \left(\int_{\mathbb{R}^n} |f(u(x,s)) - f(0)|^2 dx \right)^{\frac{1}{2}} \\
&\le \sup_{0<s<T} \left(\int_{\mathbb{R}^n} |f'(\zeta_x(x,s))|^2 |u(x,s)|^2 dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$\le ||f'||_\infty \sup_{0<s<T} \left(\int_{\mathbb{R}^n} |u(x,s)|^2 dx \right)^{\frac{1}{2}}$$
$$= ||f'||_\infty ||u||_{\mathcal{X}_T^{(2)}},$$

where $\zeta_x(x,s)$ is an intermediate point between $u(x,t)$ and 0. □

Lemma 3.8. *Let f be as in Lemma 3.7 and let $\mathcal{N}$ be defined by (3.11). Then we have*

$$||\mathcal{N}(u)||_{\mathcal{X}_T^{(2)}} \le ||g||_{\mathcal{X}_T^{(2)}} + 2c_0 T^{\frac{1}{2}} ||f'||_\infty ||u||_{\mathcal{X}_T^{(2)}}. \tag{3.13}$$

Proof. For $u \in \mathcal{X}_T^{(2)}$ we have, using the fact that T_t is an L^2-contraction,

$$||\mathcal{N}(u)||_{\mathcal{X}_T^{(2)}} = ||T_t g - \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(u) ds||_{\mathcal{X}_T^{(2)}}$$
$$\le \sup_{0<s<T} \left(||g||_{L^2} + || \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(u) ds||_{L^2} \right).$$

Using (3.8) to estimate the second term in the last line we then have

$$||\mathcal{N}(u)||_{\mathcal{X}_T^{(2)}} \le ||g||_{L^2} + \sup_{0<s<T} \int_0^t c_0 (t-s)^{-\frac{1}{2}} ||f(u)||_{L^2} ds$$
$$\le ||g||_{L^2} + c_0 \left(\sup_{0<t<T} 2t^{\frac{1}{2}} \right) \sup_{0<s<T} ||f(u(\cdot,s))||_{L^2}$$
$$= ||g||_{L^2} + 2c_0 T^{\frac{1}{2}} ||f(u)||_{\mathcal{X}_T^{(2)}}$$
$$\le ||g||_{L^2} + 2c_0 T^{\frac{1}{2}} ||f'||_\infty ||u||_{\mathcal{X}_T^{(2)}}.$$

The last inequality follows from (3.12). (Note the triviality $||g||_{L^2} = ||g||_{\mathcal{X}_T^{(2)}}$, since g is independent of t). □

Lemma 3.9. *The operator $\mathcal{N}$ satisfies*

$$||\mathcal{N}(u) - \mathcal{N}(v)||_{\mathcal{X}_T^{(2)}} \le 2c_0 T^{\frac{1}{2}} ||f'||_\infty ||u - v||_{\mathcal{X}_T^{(2)}}. \tag{3.14}$$

Proof. From the definition of $\mathcal{N}$ it follows that
$||\mathcal{N}(u) - \mathcal{N}(v)||_{\mathcal{X}_T^{(2)}} =$

$$||T_t g - \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(u) ds - T_t g + \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} f(v) ds||_{\mathcal{X}_T^{(2)}}$$
$$= \sup_{0<s<T} || \int_0^t [q(x,D)]^{\frac{1}{2}} T_{t-s} (f(u) - f(v)) ds||_{L^2}$$

$$
\begin{aligned}
&\leq \sup_{0<s<T} \int_0^t \frac{c_0}{(t-s)^{\frac{1}{2}}} ||f(u) - f(v)||_{L^2} ds \\
&\leq 2c_0 T^{\frac{1}{2}} \sup_{0<s<T} ||f(u) - f(v)||_{L^2} \\
&\leq 2c_0 T^{\frac{1}{2}} ||f'||_\infty ||u - v||_{\mathfrak{X}_T^{(2)}}
\end{aligned}
$$

where the first inequality follows from (3.8) and the last inequality follows from (3.12). □

Lemma 3.10. *Let f be as in Lemma 3.7 and $g \in \mathfrak{X}_T^{(2)}$. Let $R \in \mathbb{R}$ be such that*

(3.15) $||g||_{\mathfrak{X}_T^{(2)}} \leq \dfrac{R}{4}.$

For

(3.16) $T \leq \dfrac{1}{25 c_0^2 ||f'||_\infty^2},$

it follows that the ball

(3.17) $\bar{B}(g; R) := \{v \in \mathfrak{X}_T^{(2)} : ||g - v||_{\mathfrak{X}_T^{(2)}} \leq R\}$

is invariant under $\mathcal{N}$.

Proof. For $u \in \bar{B}(g; R)$ it follows from Lemma 3.8 and the triangle inequality that

$$||\mathcal{N}(u) - g||_{\mathfrak{X}_T^{(2)}} \leq 2||g||_{\mathfrak{X}_T^{(2)}} + 2c_0 T^{\frac{1}{2}} ||f'||_\infty ||u||_{\mathfrak{X}_T^{(2)}}$$

and we also have

$$||u||_{\mathfrak{X}_T^{(2)}} \leq ||g||_{\mathfrak{X}_T^{(2)}} + R.$$

Then using (3.15) and (3.16) we get

$$
\begin{aligned}
||\mathcal{N}(u) - g||_{\mathfrak{X}_T^{(2)}} &\leq 2||g||_{\mathfrak{X}_T^{(2)}} + 2c_0 T^{\frac{1}{2}} ||f'||_\infty (||g||_{\mathfrak{X}_T^{(2)}} + R) \\
&\leq \frac{R}{2} + 2c_0 T^{\frac{1}{2}} ||f'||_\infty \frac{5R}{4} \\
&\leq R
\end{aligned}
$$

i.e. $u \in \bar{B}(g; R)$ implies $\mathcal{N}(u) \in \bar{B}(g; R)$. □

Lemma 3.9 shows that $\mathcal{N}$ is a contraction mapping on ${\mathfrak{X}_T}^{(2)}$ provided that $T < \dfrac{1}{4c_0^2 ||f'||_\infty^2}$. Lemma 3.10 shows that there exists a nonempty closed subset of $\mathfrak{X}_T^{(2)}$ that is invariant under $\mathcal{N}$. Using both we finally arrive, via the Banach contraction theorem, at:

Theorem 3.11. *Let $f \in C(\mathbb{R})$ have bounded, continuous derivative f' and assume $f(0) = 0$. Suppose also that $-q(x, D)$ generates an analytic L^2-sub-Markovian semigroup and in particular assume that* (3.8) *holds for some $c_0 > 0$. Then for given $g \in L^2(\mathbb{R}^n)$ there exists a unique mild solution to* (3.9) *in the interval $(0, T)$ if*

$$T \leq \min\left(\frac{1}{4c_0^2||f'||_\infty^2}, \frac{1}{25c_0^2||f'||_\infty^2}\right) = \frac{1}{25c_0^2||f'||_\infty^2}.$$

□

This theorem ensures, at least locally with respect to t, the existence of a mild solution to the evolution equation for certain non-linear pseudo-differential operators. This is a completely new result. It has some straightforward extensions: Since (3.8) is a special case of

$$||[q(x,D)]^\alpha T_t|| \leq \tilde{c}_0 t^{-\alpha}, \qquad 0 < \alpha < 1, \tag{3.18}$$

we may derive a similar result for the equation

$$\left\{ \begin{array}{rcl} u_t + q(x,D)u + [q(x,D)]^\alpha f(u) &=& 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) &=& g(x) \text{ on } \mathbb{R} \times \{t=0\}. \end{array} \right. \tag{3.19}$$

In particular, in the case where for some $\alpha \in (0,1)$ we have

$$||[q(x,D)]^\alpha u||_{L^2} \sim ||\nabla u||_{L^2} \tag{3.20}$$

we may compare the problem

$$\left\{ \begin{array}{rcl} u_t + \epsilon q(x,D)u + [q(x,D)]^\alpha f(u) &=& 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) &=& g(x) \text{ on } \mathbb{R} \times \{t=0\} \end{array} \right. \tag{3.21}$$

with the problem

$$\left\{ \begin{array}{rcl} u_t + \epsilon q(x,D)u + b \cdot \nabla f(u) &=& 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) &=& g(x) \text{ on } \mathbb{R} \times \{t=0\} \end{array} \right.$$

for $b \in \mathbb{R}^n$. The main task would then be to control the commutator

$$[q(x,D), \frac{\partial}{\partial x_j}]u = q(x,D)\frac{\partial u}{\partial x_j} - \frac{\partial}{\partial x_j}\left(q(x,D)u\right).$$

This requires precise estimates within a symbolic calculus for $q(x, D)$ which is a topic beyond the investigations of this thesis.

Notation

$Du, \nabla u$ gradient vector of u;
$\nabla \cdot u$ divergence of u;
∂U boundary of the region U;
H^* Legendre transform of the operator H;
Z^* dual space of the normed space Z;
$|\alpha|$ $\alpha_1 + \dots \alpha_n$ when α is a multi-index;
$\hat{u}$ Fourier transform of u (alternative notation Fu);
$\bar{u}$ complex conjugate of u;
$(\, , \,)_0$ L^2- inner product;
$(\, | \,)_H$ inner product in the space H;
$< \, , \, >$ duality pairing between a space and its dual;
$< \, , \, >_V$ duality pairing between V and its dual space V^*;
$|| \cdot ||_X$ norm corresponding to the space X;
$|| \cdot ||_\infty$ $|| \cdot ||_{L^\infty}$;
$|| \cdot ||$ norm where the space is obvious;
$R(A)$ range of the operator A;
∂_x^k $\frac{\partial^k}{\partial x}$ for $k \in \mathbb{N}$;
$D^\alpha u(x)$ $\frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ or $\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x)$, for a multi-index α;
$\triangle$ Laplace operator;
$V \subset\subset U$ $V \subset \bar{V} \subset U$, where U, V are open subsets of $\mathbb{R}^n$and $\bar{V}$ is compact;
$C^k(G)$ k-times continuously differentiable functions on the space G;
$C^\infty(G)$ infinitely often differentiable continuous functions on the space G;
$C_0(G)$ continuous functions on the space G with compact support;
$C_b(G)$ bounded continuous functions on the space G;
$L^p, p \in [1, \infty]$ Lebesgue spaces;
$\mathcal{S}(\mathbb{R}^n)$ Schwartz space;
$\mathcal{S}'(\mathbb{R}^n)$ topological dual space of $\mathcal{S}(\mathbb{R}^n)$;
$A(\mathbb{R}^n)$ Wiener algebra on $\mathbb{R}^n$;
$\mathcal{D}'(G)$ topological dual space of C_0^∞;
$\mathcal{B}(G)$ Borel σ-field of the space G;

$\mathcal{M}_b^+(\Omega)$ set of all bounded measures on the space Ω;
$\mathcal{M}_b^1(\Omega)$ set of all probability measures on Ω;
$\mathcal{M}(G)$ set of all signed measures on G;
$\mathcal{M}_b(G)$ set of all bounded signed measures on G;
$B_b(\mathbb{R}^n;\mathbb{R})$ set of all bounded operators from $\mathbb{R}^n$ to $\mathbb{R}$;
$W^{k,p}(U)$ Sobolev spaces, $k \in \mathbb{N}$, $1 \leq p \leq \infty$;
H^s $W^{k,p}(U)$ for $p = 2$;
$B^s_{\psi,p}(\mathbb{R}^n)$ tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ s.t.
$||(1 + |\psi(\cdot)|)^{\frac{s}{2}}\hat{u}(\cdot)||_{L^p(\mathbb{R}^n)} < \infty$;
$H^{\psi,s}(\mathbb{R}^n)$ $B^s_{\psi,p}(\mathbb{R}^n)$ for $p = 2$;
$[A, B]$ $AB - BA$, commutator of operators A and B.

List of Figures

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